First order common knowledge logics

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Abstract

In this paper we investigate first order common knowledge logics; i.e., modal epistemic logics based on first order logic with common knowledge operators. It is shown that even rather weak fragments of first order common knowledge logics are not recursively axiomatizable. This applies, for example, to fragments which allow to reason about names only; that is to say, fragments the first order part of which is based on constant symbols and the equality symbols only. Then formal properties of "quantifying into" epistemic contexts are investigated. The results are illustrated by means of epistemic representations of Nash Equilibria for finite games with mixed strategies.

1 Introduction

Common knowledge logics are modal logics with one knowledge operator $\mathbf{K}_i$ for each agent (or player) i and common knowledge operators $\mathbf{C}_M$ for each set M of at least two agents. The intended meaning of $\mathbf{K}_i\varphi$ and $\mathbf{C}_M\varphi$ is "agent i knows (believes) φ" and "it is common knowledge among the agents in M that φ", respectively. Propositional common knowledge logics have been investigated intensively (cf. [3], [2], [8], [5], [1], [6]). They turned out to be useful for the epistemic analysis of games, the philosophy of knowledge, and the analysis of Multi-Agent systems in Computer Science and Artificial Intelligence.

On the other side, first order common knowledge logics have been investigated only recently, and mainly from the the proof theoretic viewpoint, see [6], [7], [11], and [12] [1]. They are, however, of interest both from the point of view of applications as well as from the viewpoint of pure logic: of course, we require first order extensions of propositional common knowledge logics whenever the application domain is infinite. An example is the epistemic analysis of the playability of games with mixed stategies, where quantification over the real numbers (or at least a real closed field) is required, see [7]. Also first order logic is required to represent knowledge (or common knowledge) about a finite domain the cardinality of which is not known in advance. This applies to most application domains in the field of Knowledge Representation, consider e.g., a domain consisting of the employees of a company, or the set of mails in a mailbox.

The following two questions on first order common knowledge logics are addressed: (1) we discuss the complexity the decision problem for various fragments of first order common knowledge logics and (2) we discuss some formal properties of "quantifying into" epistemic contexts. The first question is standard from the viewpoint of pure logic and the second question is related to the the study of "term existence theorems" in pure logic; that is to say, the question

[1] The listed papers do not investigate first order common knowledge logic directly but infinitary first order modal logics. Of course, first order common knowledge logics are fragments of the infinitary ones, since $\mathbf{C}$ can be expressed using an infinite conjunction.

whether an existential quantifier can be replaced with a term. However, both problems are relevant to possible applications of first order common knowledge logics.

- From the viewpoint of its applicability in Knowledge Representation the decision problem is of course fundamental. Applicable systems require the design of an inference engine and so the logic should be decidable or at least recursively enumerable. But also from a more theoretical viewpoint—e.g., for the epistemic analysis of games—the decidability (or recursive enumerability) of a logic of knowledge is of interest: for example, if a common knowledge logic L is not recursively enumerable then (for all the cases considered here), also the set

  $$\{\psi : \mathbf{K}_1\psi \in L\}$$

  is not recursively enumerable. That is to say, the set of sentences an agent would know without making any assumptions about the world is not recursively enumerable. The agent would—in a sense—be considerably more intelligent than a Turing machine which might be a reason to reject L as a reasonable system of knowledge for certain applications. In general, results about the decision problem for a logic of knowledge show the implicit computational abilities of the agents one assumes by applying a certain logic of knowledge (or belief).

  It will be shown that semantically defined full first order common knowledge logics are in fact highly undecidable. In contrast to the (easily proved) fact that all the standard modal epistemic logics without common knowledge operators are finitely axiomatizable (and so recursively enumerable), the set of valid formulas of every reasonable semantically defined first order common knowledge logic is not recursively enumerable. We show that even

  - the fragments with monadic predicates only (without equality and function symbols) and
  - the fragments with constant symbols and equality (without predicate symbols and proper function symbols) only

  are not recursively axiomatizable. Observe that these negative results show that the axiomatization results for infinitary modal predicate logics in [11] are the best we can do: there is no finite (alias effective) way to axiomatize first order common knowledge logics, so a non-effective axiomatization in infinitary logic is an interesting alternative.

  Finally, we will briefly describe a possibility to obtain decidable fragments.

- A language allows to "quantify into" epistemic contexts iff it allows to apply epistemic operators to open formulas. The possibility to "quantify into" epistemic contexts is the main reason for the expressive power of first order common knowledge logics. In fact, first order common knowledge languages which allow epistemic operators to be applied to closed formulas only are just simple combinations of propositional common knowledge logics and classical first order logic without any new interesting properties. To illustrate the importance of the possibility to "quantify into" epistemic contexts consider the following example which was developed in detail by Kaneko and Nagashima, see [6] and [7]. Consider a finite game G with mixed strategies. Denote by Γ the theory of real closed fields and by $\mathsf{Nash}_G(\overline{x})$ a formula which says that the vector $\overline{x}$ is a Nash Equilibrium for G. We have $\Gamma \models \exists\overline{x}\ \mathsf{Nash}_G(\overline{x})$. For all the common knowledge logics we consider in this paper we can deduce $\mathbf{C}\Gamma \models \mathbf{C}\exists\overline{x}\ \mathsf{Nash}_G(\overline{x})$. (Here and in what follows we denote by $\mathbf{C}\Gamma$ the set $\{\mathbf{C}\psi : \psi \in \Gamma\}$.) This means that the existence of a Nash Equilibrium for G is common knowledge whenever the theory of real closed fields is common knowledge. This does not imply, however, that there exists a Nash Equilibrium $\overline{x}$ for G which is common knowledge. That is to say, we do not necessarily have

$$\mathbf{C}\Gamma \models \exists\overline{x}\ \mathbf{C}\mathsf{Nash}_G(\overline{x}). \tag{1}$$

Whether (1) holds will turn out to depend upon the formal interpretation of the sentence *"the theory of real closed fields is common knowledge"*.

There are various contexts which give rise to the question whether we have—for a set of formulas X and a non–epistemic formula $\varphi(x)$—

$$X \models \exists x \mathbf{C}\varphi(x)\ ? \tag{2}$$

This is the easiest and probably the most interesting form of "quantifying into" epistemic contexts. In this paper we start a semantic investigation of consequences of this type.

It should be observed that most of the results presented in this paper do not depend on the behaviour of the epistemic operators but on the non–epistemic vocabulary and the interaction between quantifiers and modal operators only. So it does not make any difference for our investigation whether we model belief or knowledge or whether we assume positive introspection, negative introspection, or both.

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2 Syntax and Semantics of the full language

We define the syntax of first order common knowledge logic. The full language $\mathcal{CEL}^=$ is constructed from

- local predicate symbols $P_1, \ldots,$
- global predicate symbols $Q_1, \ldots,$
- local function symbols $f_1, \ldots,$
- global function symbols $g_1, \ldots,$
- infinitely many variables $x_1, \ldots,$
- the logical constants $\wedge$, $\Rightarrow$, $\neg$,
- the equality symbol $=$,
- a quantifier $\forall x$ for each variable x,
- knowledge operators $\mathbf{K}_1$ and $\mathbf{K}_2$,
- the common knowedge operator $\mathbf{C}$.

Observe that we assume two agents only. However, everything we prove in this paper for (fragments of) $\mathcal{CEL}^=$ is easily seen to hold for arbitrarily many agents as well. The predicate and function symbols come equipped with integers indicating their arity. Function symbols of arity 0 are called constants and predicates of arity 0 are called propositional variables. *Terms* are defined inductively as follows:

- every variable and constant is a term.
- if f is a (local or global) function symbol of arity n and $t_1, \ldots, t_n$ are terms, then $ft_1, \ldots, t_n$ is a term.

A *global term* is a term which is composed from variables and global constants by means of global function symbols. Thus variables are global terms. Put $\mathbf{P}\varphi = \neg\mathbf{C}\neg\varphi$, for any formula φ.

We briefly explain the intuitive difference between global and local symbols. (Logics will be defined semantically in this paper, so the precise definition of the meaning of global and local symbols will be given later via their interpretation in models.) The extension of a global predicate symbol as well as the interpretation of a global function symbol is assumed to be common knowledge. That is to say,

$$\forall \overline{x}(\mathbf{P}Q(\overline{x}) \Rightarrow Q(\overline{x}) \wedge Q(\overline{x}) \Rightarrow \mathbf{C}Q(\overline{x})) \tag{3}$$

is valid, for all global predicate symbols Q, and

$$\forall \overline{x} \forall y(\mathbf{P}(g(\overline{x}) = y) \Rightarrow (g(\overline{x})) = y) \wedge (g(\overline{x}) = y) \Rightarrow \mathbf{C}(g(\overline{x}) = y)) \tag{4}$$

is valid, for all global function symbols g. The equality symbol is a special global predicate symbol, and so equality between objects is assumed to be common knowledge. We will see that this simple observation is of extreme importance for formal properties of "quantifying into" epistemic contexts.

We now define the semantics for the language $\mathcal{CEL}^=$. A *state* S is a first order structure

$$S = \langle D, P_1^S \ldots, Q_1^S, \ldots, f_1^S \ldots, g_1^S, \ldots \rangle,$$

such P_i^S as well as Q_i^S are subsets of D^n (n the arity of P_i and Q_i) and f_i^S as well as g_i^S are functions from D^n into D (n the arity of f_i and g_i). We assume that 0-ary predicate symbols P just obtain a truth-value $P^S \in \{0, 1\}$.

A propositional Kripke structure is a relational structure

$$\mathcal{F} = \langle W, R_1, R_2 \rangle,$$

where W is a non-empty set of worlds and R_1, R_2 are binary relations between worlds. Propositional common knowledge logic (i.e., propositional logic extended by means of the operators $\mathbf{K}_1$, $\mathbf{K}_2$, and $\mathbf{C}$) is interpreted in propositional Kripke structures in the standard manner, see e.g. [3]. The intended meaning of $w_1 R_i w_2$ is: in world w_1 agent i believes (or knows) that w_2 is possible. Given a class of propositional Kripke frames $\mathcal{C}$ we denote by $L_p(\mathcal{C})$ the propositional common knowledge determined by $\mathcal{C}$. Various logics of the form $L_p(\mathcal{C})$ have been investigated intensively: by **K.C** we denote the propositional logic determined by the class of all propositional Kripke frames and by **K4.C**, **S4.C**, **K4D.C**, **K45D.C**, and **S5.C** we denote the propositional logic determined by the class of

- transitive,
- transitive and reflexive,
- transitive and serial,
- transitive, euclidean and serial,
- symmetric, transitive, and reflexive structures,

respectively. All these logics have been axiomatized and are known to be decidable, see [3].

We are going to combine first order structures and propositional Kripke structures in order to obtain models for first order common knowledge logic. This is done in a straightforward manner by associating with every world in a propositional Kripke structure a state representing the structure of that world: A *first order Kripke structure* $\langle \mathcal{F}, D, S \rangle$ consists of a propositional Kripke structure $\mathcal{F}$, a non-empty set of objects D and a mapping S associating with each $w \in W$ a state

$$S(w) = \langle D, P_1^{S(w)} \ldots, Q_1^{S(w)}, \ldots, f_1^{S(w)} \ldots, g_1^{S(w)}, \ldots \rangle,$$

such that $Q_i^{S(w)} = Q_i^{S(v)}$ and $g_i^{S(w)} = g_i^{S(v)}$, for all $w, v \in W$ and $i \in \omega$. This reflects the fact that the interpretation of global symbols does not vary between worlds. In the situation above we say that $\langle \mathcal{F}, D, S \rangle$ is based on $\mathcal{F}$. Observe that we assume constant domains; that is to say, the domain D is not allowed to vary from state to state. This assumption means that the domain is common knowledge and is syntactically reflected by the validity of the Barcan formula:

$$\forall \overline{x} \mathbf{C} \varphi \Leftrightarrow \mathbf{C} \forall \overline{x} \varphi.$$

A valuation $\mathcal{V}$ into $\langle \mathcal{F}, D, S \rangle$ is a mapping from the set of inividual variables into D. Define by induction the interpretation $t^{S(w)}$ of a term t in a world w and the relation $\models_{\mathcal{V}}$ between worlds and formulas:

- for all variables x: $x^{S(w)} = \mathcal{V}(x)$.
- for all function symbol f of arity n and terms $t_1, \dots, t_n$,

$$f(t_1, \dots, t_n)^{S(w)} = f^{S(w)}(t_1^{S(w)}, \dots, t_n^{S(w)}).$$

- $w \models_{\mathcal{V}} t_1 = t_2$ iff $t_1^{S(w)} = t_2^{S(w)}$, for all terms t_1, t_2.
- $w \models_{\mathcal{V}} P(t_1, \dots, t_n)$ iff $\langle t_1^{S(w)}, \dots, t_n^{S(w)} \rangle \in P^{S(w)}$.
- the steps for conjunction, negation, implication, and quantification are as usual.
- $w \models_{\mathcal{V}} \mathbf{K}_i \varphi$ iff $v \models \varphi$, for all v with wR_iv, for $i \in \{1, 2\}$.
- $w \models_{\mathcal{V}} \mathbf{C} \varphi$ iff $v \models \varphi$, for all v with $w(R_1 \cup R_2)^* v$. $(R_1 \cup R_2)^*$ is the reflexive and transitive closure of $R_1 \cup R_2$.

A closed formula φ is valid in a structure $\mathcal{G}$, in symbols $\mathcal{G} \models \varphi$, if $w \models \varphi$, for every world w in $\mathcal{G}$.

For a class of propositional Kripke structures $\mathcal{C}$ we denote by $L(\mathcal{C})$ the set of closed formulas valid in all first order Kripke structures based on structures in $\mathcal{C}$. Given a propositional logic $L = L_p(\mathcal{C})$—where $\mathcal{C}$ is the class of all propositional Kripke structures validating L—we put $L^Q = L(\mathcal{C})$. So $\mathbf{K}.\mathbf{C}^Q$ is the logic determined by the class of all first order Kripke structures. Let $L = L(\mathcal{C})$, Γ a set of formulas, and φ a formula. Then φ follows from Γ in L, in symbols $\Gamma \models_L \varphi$, iff for all $\mathcal{F} \in \mathcal{C}$, all first order Kripke structures $\langle \mathcal{F}, D, S \rangle$ based on $\mathcal{F}$, all valuations $\mathcal{V}$, and all worlds w:

$$w \models_{\mathcal{V}} \Gamma \text{ implies } w \models_{\mathcal{V}} \varphi.$$

We now define the notion of a (standard) first order common knowledge logic. The definition was chosen for technical reasons and we do not claim that it covers all reasonable first order common knowledge logics. It covers, however, all the popular candidates.

DEFINITION 2.1 (First order common knowledge logic) A set $L \subseteq \mathcal{CEL}^{=}$ is called a first order common knowledge logic iff there exists a class $\mathcal{C}$ of propositional Kripke frames containing all frames of the form $\langle W, R_1, R_2 \rangle$, R_1 and R_2 equivalence relations, such that $L = L(\mathcal{C})$.

The class $\mathcal{STD}$ of standard first order common knowledge logics is defined by putting

$$\mathcal{STD} = \{\mathbf{K}.\mathbf{C}^Q,\ \mathbf{K4}.\mathbf{C}^Q,\ \mathbf{S4}.\mathbf{C}^Q,\ \mathbf{K4D}.\mathbf{C}^Q,\ \mathbf{K45D}.\mathbf{C}^Q,\ \mathbf{S5}.\mathbf{C}^Q\ \}.$$

All the results we are going to formulate will either hold for the class of all first order common knowledge logics or for the class of all standard ones. Hence they do not depend on the specific properties of the knowledge operators.

3 An example

We formalize a 2–person finite strategic–form game G with mixed strategies. Recall that such a game consists of two finite sets $S_1 = \{s_1^1, \ldots, s_{l(1)}^1\}$ and $S_2 = \{s_2^2, \ldots, s_{l(2)}^2\}$ of strategies and payoff functions u_i, $i = 1, 2$, from $S = S_1 \times S_2$ into the set of rational numbers. $u_i(s_1, s_2)$ is the payoff of player i from the pure strategy combination $\langle s_1, s_2 \rangle \in S_1 \times S_2$.

G can be represented on the basis of the language $\mathcal{L}_R$ of real closed fields. Recall that this consists of

- function symbols: 0^r, 1^r, $+^r$, $-^r$, $\cdot^r$, $/^r$,
- predicate symbols: $\leq^r$.

The intended meaning of those symbols should be clear. We shall mostly omit the superscript r. The theory of real closed fields is complete. Hence it coincides with the theory of the real numbers. Observe that we did not fix yet whether the symbols are global or local. Of course, in classical logic this distinction is meaningless. To formalize G suppose that

- $[u_i(s_1, s_2)]$, for $s_1, s_2 \in S_1 \times S_2$ and $i \in \{1, 2\}$,

are closed terms. They are used to denote the payoff for player i from the pure strategy combination $\langle s_1, s_2 \rangle$. A *mixed strategy* of player i is a vector of terms $\bar{t}_i = \langle t_{i1}, \ldots, t_{il(i)} \rangle$ such that

$$\mathsf{St}(\bar{t}_i) = (\sum_{k=1}^{l(i)} t_{ik} = 1) \wedge (\bigwedge \{t_{ik} \geq 0 : 1 \leq k \leq l(i)\}).$$

The *expected payoff* to player i from a mixed strategy combination $\bar{t} = \langle \bar{t}_1, \bar{t}_2 \rangle$ is given by the term

$$G_i(\bar{t}) = \sum_{k_1} \sum_{k_2} t_{1k_1} \cdot t_{2k_2} \cdot [u_i(s_{k_1}^1, s_{k_2}^2)].$$

Now we have all the basic ingredients required to formalize G. For example, given the mixed strategy combination $\overline{t} = \langle \overline{t}_1, \overline{t}_2 \rangle$ we define

$$\begin{aligned}\mathsf{Nash}_G(\overline{t}) \quad = \quad & \mathsf{St}(\overline{t}_1) \wedge \forall \overline{x}_1 (\mathsf{St}(\overline{x}_1) \Rightarrow (G_1(\overline{t}) \geq G_1(\overline{x}_1, \overline{t}_2)) \\ & \wedge \mathsf{St}(\overline{t}_2) \wedge \forall \overline{x}_2 (\mathsf{St}(\overline{x}_2) \Rightarrow G_2(\overline{t}) \geq G_2(\overline{x}_2, \overline{t}_1)).\end{aligned}$$

This formula says that $\overline{t}$ is a Nash Equilibrium for G. Denote by Φ_r an axiomatization of the theory of real closed fields (with the equality symbol $=$). Then

$$\Phi_r \models \exists \overline{x} \mathsf{Nash}_G(\overline{x})$$

holds for every game G. This follows from the facts that (1) every game has a Nash Equilibrium in the real numbers (see [9] and [14]) and (2) the theory of real closed fields is complete (see [13]). So far we did not use the epistemic part of the language. This is required, however, to give an epistemic analysis of the game. In what follows we assume that an axiomatization Φ_r of the theory of real closed fields is common knowledge and ask for various sentences φ whether they follow (in a logic L) from $\mathbf{C}\Phi_r$ or not. Of course, we have for any finite game G:

$$\mathbf{C}\Phi_r \models_L \mathbf{C}\exists \overline{x} \mathsf{Nash}_G(\overline{x});$$

that is to say, it is common knowledge that for every game there exists a Nash Equilibrium. When does it follow that there exists a vector $\overline{x}$ such that it is common knowledge that $\overline{x}$ is a Nash Equilibrium for G?

It turns out that this depends on whether the function and predicate symbols involved are global or local. Given a propositional frame $\langle W, R_1, R_2 \rangle$ and $w \in W$ we say that $v \in W$ is *reachable* from w if $w(R_1 \cup R_2)^* v$. Now we have the following cases:

- Suppose all function and predicate symbols in Φ_r and $\mathsf{Nash}_G(\overline{x})$ are local symbols. Then $w \models \mathbf{C}\Phi_r$ means that all the worlds v reachable from w are real closed fields with the same domain, but that the interpretation of the symbols may vary between the worlds. In this case $\mathbf{C}\Phi_r \models_L \exists \overline{x} \mathbf{C} \mathsf{Nash}_G(\overline{x})$ holds iff $\Phi_r \models \forall \overline{x} \mathsf{Nash}_G(\overline{x})$ holds in classical logic. (This follows from Theorem 4.1 below.) Thus, under the local interpretation of function and predicate symbols there are no interesting games the Nash Equilibria of which are common knowledge. One explanation seems to be that the constants $[u_i(s_1, s_2)]$ are local. Hence the players do not know the payoff functions of the game and so we should not expect them to know the Nash Equilibria for the game. However, it is not difficult to extend the technique of the proof of Theorem 4.1 to show that the same result holds under the assumption that the constants $[u_i(s_1, s_2)]$ are global.

- Suppose all function symbols in Φ_r and $\mathsf{Nash}_G(\overline{x})$ are interpreted globally. Then $\leq^r$ is also interpreted globally, as is easy to see. Then $w \models \mathbf{C}\Phi_r$

means that all the worlds v reachable from w actually coincide and obviously $\mathbf{C}\Phi_r \models_L \exists\overline{x}\mathbf{C}\mathsf{Nash}_G(\overline{x})$ holds for all L. In this case "quantifying into" epistemic contexts is trivial.

- Again we assume that all function symbols in Φ_r and $\mathsf{Nash}_G(\overline{x})$ are interpreted globally. However, this time we do not use the equality symbol to express equality, but we use a new (local) predicate symbol eq to express equality. Denote by φ^{eq} the formula which results from φ when all occurences of the equality symbol in φ are replaced with eq. Denote by $A(\mathsf{eq})$ a set of formulas axiomatizing equality in first order logic. Then

$$\mathbf{C}(\Phi_r^{\mathsf{eq}} \cup A(\mathsf{eq})) \models_L \exists\overline{x}\mathbf{C}\mathsf{Nash}_G^{\mathsf{eq}}(\overline{x})$$

  holds iff there exists a vector of closed terms $\overline{t}$ such that

$$\mathbf{C}(\Phi_r^{\mathsf{eq}} \cup A(\mathsf{eq})) \models_L \mathbf{C}\mathsf{Nash}_G(\overline{t}),$$

  see Theorem 4.2 below. This case was considered in a proof theoretic manner in [7].

Thus various interesting properties of the resulting system depend on whether local or global symbols are chosen. If symbols of mathematical theories are assumed to be local, then most of the resulting properties of "quantifying into" epistemic contexts can be explained by the fact that mathematics is common knowledge only in a very weak sense. It seems that, in order to model a situation in which mathematics is common knowledge—but certain facts in the non-mathematical world are not—all the symbols of the background mathematical theory should be global. In conclusion, the results of [6] and [7] should be interpreted as results about a particular interpretation of the concept of common knowledge about real closed fields and not as results about common knowledge in game theory.

We illustrate the situation when global symbols are used for the underlying mathematics by means of the following example.

Assume that all the symbols of Φ_r and $\mathsf{Nash}_G(\overline{x})$ are global. Introduce new vectors of local constants $\overline{a}_i = \langle a_{i1}, \ldots, a_{il(i)} \rangle$, for $i = 1, 2$. $\overline{a}_i$ is intended to denote the mixed strategy chosen by player i. To express that player i knows his own mixed strategy the following abbreviation is required.

Let Y_1 and Y_2 be sets of local predicate and function symbols, respectively. Let, for $i \in \{1, 2\}$, $\mathsf{Glob}_i(Y_1, Y_2)$ be the union of the following two sets:

$$\{\forall\overline{x}(\neg\mathbf{K}_i\neg P(\overline{x}) \Rightarrow P(\overline{x}) \wedge P(\overline{x}) \Rightarrow \mathbf{K}_i P(\overline{x}) : P \in Y_1\},$$

$$\{\forall\overline{x}\forall y(\neg\mathbf{K}_i\neg f(\overline{x}) = y \Rightarrow f(\overline{x})) = y \wedge f(\overline{x}) = y \Rightarrow \mathbf{K}_i f(\overline{x}) = y : f \in Y_2\}.$$

It should be clear that $\mathsf{Glob}_i(Y_1, Y_2)$ says that the extensions of the symbols in Y_1, Y_2 are known by player i; it does not follow, however, that they are common knowledge. Denote by Γ the following set:

- Φ_r, $\mathsf{Glob}_1(\emptyset, \{\overline{a}_1\})$, $\mathsf{Glob}_2(\emptyset, \{\overline{a}_2\})$,
- $\forall \overline{y}_1(\mathsf{St}(\overline{y}_1) \Rightarrow (G_1(\overline{a}) \geq G_1(\overline{y}_1, \overline{a}_2))$, $\forall \overline{y}_2(\mathsf{St}(\overline{y}_2) \Rightarrow (G_2(\overline{a}) \geq G_2(\overline{a}_1, \overline{y}_2))$.
- $\mathsf{St}(\overline{a}_1)$, $\mathsf{St}(\overline{a}_2)$.

(Here and in what follows $\overline{a} = \langle \overline{a_1}, \overline{a_2} \rangle$.) For example, $\mathbf{C}\mathsf{Glob}_i(\emptyset, \{\overline{a_i}\})$ means that it is common knowledge that player i knows $\overline{a_i}$. Of course we have $\mathbf{C}\Gamma \models_L \mathbf{C}\mathsf{Nash}_G(\overline{a})$. But we do not have $\mathbf{C}\Gamma \models_L \exists \overline{x} \mathbf{C} \overline{x}(\overline{x} = \overline{a})$, i.e., the final decision $\overline{a}$ is not common knowledge. A counterexample can be constructed using a game G with two Nash equilibria $\langle \overline{a}_1, \overline{a}_2 \rangle$ and $\langle \overline{a}_1, \overline{a}_2' \rangle$ with $\overline{a}_2 \neq \overline{a}_2'$. The interpretation of this result is as follows: the two players agree upon a name $\overline{a}$ for a Nash Equilibrium for G but the denotation of that name is not common knowledge.

4 Quantifying into epistemic contexts

In this section we prove various (non–epistemic) conditions equivalent to the implication $\Gamma \models_L \exists \overline{x} \mathbf{C} \varphi(\overline{x})$. To simplify notation we consider the case of only one variable x in $\overline{x}$, the general results can be proved in absolutely the same manner.

A formula (a set of formulas) is called *non–epistemic* if it does not contain epistemic operators. A formula is *without interaction* (of quantifiers and epistemic operators) if it does not contain a subformula of the form $\mathbf{K}_i \psi$ or $\mathbf{C} \psi$ such that ψ is open.

In what follows we assume that Γ is a set of closed formulas and $\varphi(x)$ is a formula containing the free variable x only. The following theorem states that no non–trivial assertion of the form $\exists x \mathbf{C} \varphi(x)$ follows from a theory without interaction and global symbols. Moreover, even the addition of a weak form of "quantifying into" epistemic contexts does not change this situation. Namely, we allow that certain symbols are global relative to one of the players; that is to say, it is common knowledge that some players know their extensions but their extensions are not common knowledge.

THEOREM 4.1 *(I) Let L be a first order common knowledge logic and suppose φ and Γ are non-epistemic and do not contain global function symbols nor global predicate symbols (but possibly the equality symbol). Suppose that Y_1^1, Y_1^2 are disjoint sets of local predicate symbols which do not occur in Γ and that Y_2^1, Y_2^2 are disjoint sets of local function symbols which do not occur in Γ. Then the following conditions are equivalent:*

- $\mathbf{C}\Gamma \models_L \exists x \mathbf{C} \varphi(x)$.
- $\mathbf{C}(\Gamma \cup \mathsf{Glob}_1(Y_1^1, Y_2^1) \cup \mathsf{Glob}_2(Y_1^2, Y_2^2)) \models_L \exists x \mathbf{C} \varphi(x)$.
- $\Gamma \models \forall x \varphi(x)$ *holds in classical first order logic.*

- $\mathbf{C}\Gamma \models_L \forall x \mathbf{C}\varphi(x)$.
- *For all terms t:* $\mathbf{C}\Gamma \models_L \mathbf{C}\varphi(t)$.

(II) Suppose $L \in \mathcal{STD}$, φ is non–epistemic, Γ is closed and without interaction, and both do not contain global function symbols nor global predicate symbols. Then the following conditions are equivalent:

- $\Gamma \models_L \exists x \mathbf{C}\varphi(x)$.
- $\Gamma \models_L \forall x \mathbf{C}\varphi(x)$.
- *For all closed terms t:* $\mathbf{C}\Gamma \models_L \mathbf{C}\varphi(t)$.

Proof In this proof global symbols (save the equality symbol) are not required. So it suffices to take first order structures which interpret local symbols only. We require the following definition. Suppose

$$S = \langle D, P_1^S, \ldots, f_1^S, \ldots \rangle$$

is a first order structure and $a \in D$. Then we define, for $b \in D$, a first order structure S_{ab} as follows. Let $\sigma : D \to D$ be the mapping defined by putting $\sigma(d) = d$, for all $d \in D - \{a, b\}$, $\sigma(a) = b$, $\sigma(b) = a$. Thus σ replaces a by b and vice versa. Define $P_i^{S_{ab}}$ and $f_i^{S_{ab}}$, $i \in \omega$, by putting for all $d_1, \ldots, d_k, d \in D$ and all P_i and f_i of arity k,

$$P_i^{S_{ab}}(\sigma(d_1), \ldots, \sigma(d_k)) \text{ iff } P_i^S(d_1, \ldots, d_k),$$

and

$$f_i^{S_{ab}}(\sigma(d_1), \ldots, \sigma(d_k)) = \sigma(d) \text{ iff } f_i^S(d_1, \ldots, d_k) = d.$$

Now let

$$S_{ab} = \langle D, P_1^{S_{ab}} \ldots, f_1^{S_{ab}}, \ldots \rangle.$$

Proof of (I). (1) $\Rightarrow$ (2), (3) $\Rightarrow$ (4), (4) $\Rightarrow$ (5), and (5) $\Rightarrow$ (1) are obvious. It remains to show (2) $\Rightarrow$ (3). Assume that $\forall x \varphi(x)$ does not follow (in classical logic) from Γ. We are going to prove that (2) does not hold. Assume for simplicity that $Y_1^1 = \{h_1\}$, $Y_2^1 = \{L_1\}$, $Y_1^2 = \{h_2\}$, and $Y_2^2 = \{L_2\}$. By $P_1, \ldots$ and $f_1, \ldots$ we denote the remaining local symbols.

Take a first order structure

$$S = \langle D, P_1^S, \ldots, L_1^S, L_2^S, f_1^S \ldots, h_1^S, h_2^S \rangle$$

such that $S \models \Gamma$ but $S \not\models \forall x \varphi(x)$. Take $a \in D$ such that, for a valuation $\mathcal{V}$ with $\mathcal{V}(x) = a$, we have $S \models_{\mathcal{V}} \neg\varphi(x)$. We may assume (using Löwenheim–Skolem) that $D = \{b(0), b(1), \ldots\}$.

For each $b \in D$ define a valuation $\mathcal{V}_b$ in S_{ab} by putting $\mathcal{V}_b(x) = b$. Obviously

$$S_{ab} \models_{\mathcal{V}_b} \neg\varphi(x).$$

Define $\langle W, R_1, R_2 \rangle, D, S' \rangle$ by putting

- $W = \{0, 1, \ldots\}(= \omega)$,
- $R_1 = \bigcup\{\{2n, 2n+1\} \times \{2n, 2n+1\} : n \in \omega\}$,
- $R_2 = \bigcup\{\{2n+1, 2n+2\} \times \{2n+1, 2n+2\} : n \in \omega\}$,
- $S'(2n) = S_{ab(n)}$, for all $n \in \omega$,
-
$$\begin{aligned} S'(2n+1) \quad = \quad & \langle D, P_1^{S'(2n+1)}, \ldots, L_1^{S'(2n+1)}, L_2^{S'(2n+1)}, f_1^{S'(2n+1)}, \ldots, \\ & h_1^{S'(2n+1)}, h_2^{S'(2n+1)}\rangle, \end{aligned}$$

  for all $n \in \omega$, where

  - $P_i^{S'(2n+1)} = P_i^{S_{ab(n)}}$, for all i,
  - $L_1^{S'(2n+1)} = L_1^{S_{ab(n)}}$, $L_2^{S'(2n+1)} = L_2^{S_{ab(n+1)}}$,
  - $f_i^{S'(2n+1)} = f_i^{S_{ab(n)}}$, for all i,
  - $h_1^{S'(2n+1)} = h_1^{S_{ab(n)}}$, $h_2^{S'(2n+1)} = h_2^{S_{ab(n+1)}}$.

It remains to observe that $0 \models \mathbf{C}(\Gamma \cup \mathsf{Glob}_1(Y_1^1, Y_2^1) \cup \mathsf{Glob}_2(Y_1^2, Y_2^2))$ but that $0 \not\models \exists x \mathbf{C}\varphi(x)$.

Proof of (II). (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) are obvious. It remains to show (1) $\Rightarrow$ (2). Suppose we do not have $\Gamma \models_L \forall x \mathbf{C}\varphi(x)$. Take a first order Kripke structure $\langle\langle W, R_1, R_2\rangle, D, S\rangle$ validating L such that $w \models \Gamma$ but $w \not\models \forall x \mathbf{C}\varphi(x)$. Take a v which is reachable from w such that $v \models_{\mathcal{V}} \neg\varphi(x)$, where $\mathcal{V}(x) = a$. We define a first order Kripke structure $\langle\langle W', R_1', R_2'\rangle, D, S'\rangle$ by putting $W' = W \cup D$,

$$w_1 R_i' w_2 \Leftrightarrow \begin{cases} w_1 R_i w_2 & : \quad w_1, w_2 \in W, \\ v R_i w_2 & : \quad w_1 \in D, w_2 \in W, \\ w_1 R_i v & : \quad w_2 \in D, w_1 \in D, \\ v R_i v & : \quad w_1, w_2 \in D, \end{cases}$$

$$S'(w) = \begin{cases} S(w) & : \quad w \in W, \\ S_{ab} & : \quad w = b \in D. \end{cases}$$

It is not difficult to check that $\langle\langle W', R_1', R_2'\rangle, D, S'\rangle$ validates L whenever $L \in \mathcal{STD}$. Moreover, $w \models' \Gamma$ but not $w \models' \exists x \mathbf{C}\varphi(x)$, where $\models'$ is the satisfaction relation in the new model. ❑

In the following theorem we allow global function symbols but no equality symbol. In this case non-trivial assertions of the form $\exists x \mathbf{C}\varphi(x)$ follow from theories without (explicit) "quantifying into" epistemic contexts; namely, $\exists x \mathbf{C}\varphi(x)$ follows from a theory iff there is a global term t such that $\varphi(t)$ follows from that theory. A similar result was obtained in a proof theoretic manner in [6].

THEOREM 4.2 *Let L be a first order common knowledge logic and suppose that φ and Γ are non-epistemic and do not contain equality nor global predicate symbols. Suppose that Y_1^1, Y_1^2 are disjoint sets of local predicate symbols which do not occur in Γ and that Y_2^1, Y_2^2 are disjoint sets of local function symbols which do not occur in Γ. Then the following conditions are equivalent:*

- $\mathbf{C}\Gamma \models_L \exists x \mathbf{C}\varphi(x)$.
- $\mathbf{C}(\Gamma \cup \mathsf{Glob}_1(Y_1^1, Y_2^1) \cup \mathsf{Glob}_2(Y_1^2, Y_2^2)) \models_L \exists x \mathbf{C}\varphi(x)$.
- *there exists a global term t such that $\Gamma \models \varphi(t)$ holds in classical logic.*
- *there exists a global term t such that $\mathbf{C}\Gamma \models_L \mathbf{C}\varphi(t)$.*

Proof (1) $\Rightarrow$ (2), (3) $\Rightarrow$ (4), and (4) $\Rightarrow$ (1) are obvious. We show (2) $\Rightarrow$ (3). Assume for simplicity that $Y_1^1 = \{h_1\}$, $Y_2^1 = \{L_1\}$, $Y_1^2 = \{h_2\}$, and $Y_2^2 = \{L_2\}$. By $P_1, \ldots$ and $f_1, \ldots$ we denote the remaining local symbols and by $g_1, \ldots$ the global function symbols.

Suppose there does not exist a closed global term t such that $\Gamma \models_L \varphi(t)$. We find for each closed global term t a first order structure S_t with domain

$$D = \{s : \ s \text{ is a closed global term } \}$$

(so D does not depend on t) such that

$$g_i^{S_t}(t_1, \ldots, t_k) = g_i(t_1, \ldots, t_k),$$

for all global function symbols g_i of arity k and all closed golbal terms $t_1, \ldots, t_k$ and such that $S_t \models \Gamma$ but $S_t \not\models \varphi(t)$. (Recall that such a term model exists since the language does not contain the equality symbol.) Now the construction of a model refuting (2) is similar to the previous proof. Let $t(0), t(1), \ldots$ be an enumeration of the closed global terms.

Define $\langle W, R_1, R_2\rangle, D, S'\rangle$ by putting

- $W = \{0, 1, \ldots\}$,
- $R_1 = \bigcup\{\{2n, 2n+1\} \times \{2n, 2n+1\} : n \in \omega\}$,
- $R_2 = \bigcup\{\{2n+1, 2n+2\} \times \{2n+1, 2n+2\} : n \in \omega\}$,
- $S'(2n) = S_{t(n)}$, for all n,
- $$\begin{aligned} S'(2n+1) \quad = \quad & \langle D, P_1^{S'(2n+1)}, \ldots, L_1^{S'(2n+1)}, L_2^{S'(2n+1)}, f_1^{S'(2n+1)}, \ldots, \\ & h_1^{S'(2n+1)}, h_2^{S'(2n+1)}\rangle, \end{aligned}$$

  for all $n \in \omega$, where

- $P_i^{S'(2n+1)} = P_i^{S_{t(n)}}$, for all i,
- $L_1^{S'(2n+1)} = L_1^{S_{t(n)}}$, $L_2^{S'(2n+1)} = L_2^{S_{t(n+1)}}$,
- $f_i^{S'(2n+1)} = f_i^{S_{t(n)}}$, for all i,
- $h_1^{S'(2n+1)} = h_1^{S_{t(n)}}$, $h_2^{S'(2n+1)} = h_2^{S_{t(n+1)}}$.

Then $\langle\langle W, R_1, R_2\rangle, D, S'\rangle$ refutes (2). ❑

5 Non-axiomatizability results

We show that the monadic fragments (without equality) as well as the fragments with constant symbols and the equality symbol only of any standard common knowledge logic are not recursively axiomatizable. The proof is conducted by means of a reduction of the $\omega \times \omega$ recurrent tiling problem to the satisfaction problem for those fragments.

DEFINITION 5.1 *A tile T is a 1×1 square with fixed orientation and colored edges* $\mathsf{Right}T$, $\mathsf{Left}T$, $\mathsf{Up}T$, *and* $\mathsf{Down}T$.

Now the $\omega \times \omega$ recurrent tiling problem is defined as follows: given a finite set $\mathcal{T}$ of tiles, and a tile $T_0 \in \mathcal{T}$, is there a tiling of $\omega \times \omega$ by $\mathcal{T}$ such that T_0 occurs infinitely often in the first column. That is to say, does there exist a function f from $\omega \times \omega$ into $\mathcal{T}$ such that

- $\mathsf{Right}(f(n,m)) = \mathsf{Left}(f(n+1,m))$, *for all* $n, m \in \omega$,
- $\mathsf{Up}(f(n,m)) = \mathsf{Down}(f(n,m+1))$, *for all* $n, m \in \omega$,
- $\{n \in \omega : f(0,n) = T_0\}$ *is infinite?*

Recall that according to [4] the $\omega \times \omega$ recurrent tiling problem is Σ_1^1–complete. The reductions we provide will be quite similar to those known from the literature for propositional modal logics, see e.g. [10].

We define the formulas describing $\omega \times \omega$ recurrent tilings. First abbreviate

$$\Box\varphi = \mathbf{K}_1(\neg p \to \mathbf{K}_2(p \to \varphi)),\ \ \Diamond\varphi = \neg\Box\neg\varphi;$$

here and in what follows p denotes a 0-ary predicate symbol. Denote by R and R_0 two binary relation symbols. Given a finite set of tiles $\mathcal{T} = \{T_0, \ldots, T_M\}$ take unary relation symbols $P_0, \ldots, P_M$ and denote by $\varphi_{\mathcal{T}}$ the conjunction of the following set of formulas:

- $\mathbf{C}(p \to \forall x(\exists y(xRy) \wedge \Diamond\top))$.
- $\mathbf{C}(p \to \forall x \bigvee\{P_i(x) : 0 \le i \le M\})$.
- $\mathbf{C}(p \to \bigwedge\{P_i(x) \to \neg P_j(x) : \neg(i = j), 1 \le i, j \le M\})$.

- $\mathbf{C}(p \to \forall x(P_i(x) \to \forall y(xRy \to \bigvee\{P_j(y) : \mathsf{Right}T_i = \mathsf{Left}T_j\})))$, for $i \leq M$.
- $\mathbf{C}(p \to \forall x(P_i(x) \to \Box \bigvee\{P_j(x) : \mathsf{Up}T_i = \mathsf{Down}T_j\}))$, for $i \leq M$.
- $\exists x(p \wedge \forall y(\mathbf{P}(xR_0y) \to \exists z(\mathbf{P}(yR_0z) \wedge P_0(z)))$.

Let φ_C be the conjunction of the following set of formulas:

- $\mathbf{C}(\forall x, y(xRy \to \mathbf{C}xRy \wedge \neg(xRy) \to \mathbf{C}\neg(xRy))$,
- $\forall x \forall y(xRy \leftrightarrow xR_0y)$,
- $\mathbf{C}\forall x \forall y(xRy \to xR_0y)$,
- $\mathbf{C}\forall x \neg(xR_0x)$,
- $\mathbf{C}\forall x \forall y(\neg\mathbf{K}_1\neg(xR_0y) \to xR_0y \vee \exists z(xR_0z \wedge zR_0y))$,
- $\mathbf{C}\forall x \forall y(\neg\mathbf{K}_2\neg(xR_0y) \to xR_0y \vee \exists z(xR_0z \wedge zR_0y))$,
- $\mathbf{C}\forall x \forall y \forall z(xR_0y \wedge yR_0z \to \mathbf{CP}(xR_0z))$.

The following Lemma describes the meaning of φ_C.

LEMMA 5.2 *Suppose φ_C is satisfied in a world w_0 in $\langle \mathcal{F}, D, S\rangle$ and all worlds in $\mathcal{F}$ are reachable from w_0. Then*

- $R^{S(w)} = R^{S(w')}$, *for all $w, w' \in W$.*
- $R^{S(w_0)}$ *contains no cycle.*
- *For every $w \in W$, $\{(x, y) \in D \times D : w \models \mathbf{P}(xR_0y)\}$ is the transitive closure of $R^{S(w)}$.*

Proof Item 1 follows from the first conjunct of φ_C. Let $w \in W$ and put $T = \{(x, y) \in D \times D : w \models \mathbf{P}(xR_0y)\}$. By the third and the last conjunct, T contains the transitive closure of $R^{S(w)}$. Hence, by the fourth conjunct, $R^{S(w)}$ contains no cycle. It remains to show that T is a subset of the transitive closure of $R^{S(w)}$. But this follows inductively from the second, the fifth, and the sixth conjunct. ❑

THEOREM 5.3 *If $f : \omega \times \omega \to \mathcal{T}$ is a tiling such that $\{n \in \omega : f(0, n) = T_0\}$ is infinite, then $\varphi_\mathcal{T} \wedge \varphi_C$ is satisfiable in a first order Kripke structure validating $\mathbf{S5}.\mathbf{C}^Q$.*

Proof Suppose $f : \omega \times \omega \to \mathcal{T}$ is a tiling such that $\{n \in \omega : f(0, n) = T_0\}$ is infinite. We define a model satisfying $\varphi_\mathcal{T} \wedge \varphi_C$. To this end let

$$W = \omega \cup \mathsf{succ},$$

where

$$\mathsf{succ} = \{(n, n+1) : n \in \omega\}.$$

Let S_1, S_2 be the reflexive, transitive, and symmetrical closure of the relations S_1' and S_2', repectively, were

$$nS_1'(m,k) \text{ iff } n = m \text{ and } \mathsf{succ}(m,k),$$

and

$$(n,m)S_2'k \text{ iff } m = k \text{ and } \mathsf{succ}(n,m).$$

Define a first order Kripke structure $\langle \mathcal{F}, D, S\rangle$ based on $\mathcal{F} = \langle W, S_1, S_2\rangle$ by putting

$$D = \omega,$$

and

$$S(w) = \langle D, p^{S(w)}, R^{S(w)}, R_0^{S(w)}, P_0^{S(w)}, \ldots\rangle,$$

where

- $p^{S(w)} = 1$ iff $w \in \omega$,
- $P_i^{S(w)} = \emptyset$, for $w \in \mathsf{succ}$, and $P_i^{S(w)} = \{m \in \omega : f(m,w) = T_i\}$, for all $w \in \omega$ and $i \leq M$.
- $R^{S(w)} = \mathsf{succ}$, for all $w \in W$.
- $R_0^{S(w)} = \{(n,m) : 0 < n - m \leq 2^w\}$, for $w \in \omega$, and $R_0^{S(n,m)} = R_0^{S(n)}$, for all $(n,m) \in \mathsf{succ}$.

It is easy to see that $\varphi_{\mathcal{T}} \wedge \varphi_C$ holds in 0 in this model. ❑

THEOREM 5.4 *If $\varphi_{\mathcal{T}} \wedge \varphi_C$ is satisfiable in a first order Kripke structure then the exists a tiling $f : \omega \times \omega \to \mathcal{T}$ such that $\{n \in \omega : f(0,n) = T_0\}$ is infinite.*

Proof Suppose $\varphi_{\mathcal{T}} \wedge \varphi_C$ is satisfied in a world w_0 in a model $\langle \mathcal{F}, D, S\rangle$ with $\mathcal{F} = \langle W, S_1, S_2\rangle$ and

$$S(w) = \langle D, p^{S(w)}, R^{S(w)}, R_0^{S(w)}, P_0^{S(w)}, \ldots\rangle,$$

for $w \in W$. We may assume that all worlds in W are reachable from w_0. Let $\mathcal{V}$ be a valuation such that

$$w_0 \models_{\mathcal{V}} p \wedge \forall y(\mathbf{P}(xR_0y) \to \exists z(\mathbf{P}(yR_0z) \wedge P_0(z))).$$

By Lemma 5.2, $R^{S(w)}$ does not depend on w. Put

$$V = \{w \in W : p^{S(w)} = 1\}$$

and define a relation S on V by putting

$$v_1 S v_2 \Leftrightarrow v_1, v_2 \in V \wedge \exists w \in W - V(v_1 S_1 w S_2 v_2).$$

We may assume, by standard unravelling techniques, that S is cycle free. Take a sequence $\langle y_i : i \in \omega \rangle$ in D such that

$$\{i : y_i \in P_0^{S(w_0)}\}$$

is infinite, $\mathcal{V}(x) = y_0$ and $y_i R^{S(w_0)} y_{i+1}$, for all $i \in \omega$. It exists because $w_0 \models_{\mathcal{V}} p \wedge \forall y(\mathbf{P}(xR_0y) \to \exists z(\mathbf{P}(yR_0z) \wedge P_0(z)))$.

We find a sequence $\langle w_i : i \in \omega \rangle$ in V such that $w_i S w_{i+1}$, for all $i \in \omega$. This follows from the first conjunct of $\varphi_{\mathcal{T}}$. Now define a function f by putting $f(j,i) = T_k$, for the (uniquely determined) $k \leq M$ such that $w_i \models P_k(y_j)$. (k exists because of the second conjunct of $\varphi_{\mathcal{T}}$ and k is uniquely determined by the third conjunct.) Using the fourth and fifth conjunct it follows immediately that f defines a $\omega \times \omega$ recurrent tiling. ❑

From Theorems 5.3 and 5.4 and the Σ_1^1–completeness of the $\omega \times \omega$ recurrent tiling problem we immediately obtain:

COROLLARY 5.5 *No first order common knowledge logic is recursively axiomatizable.*

We are now going to extend this result to certain fragments:

THEOREM 5.6 *For any first order common knowledge logic L:*

- *the fragment of L with monadic predicates only (without equality and function symbols) is not recursively enumerable.*
- *The fragment of L with local constant symbols and the equality symbol only (without predicates and proper function symbols) is not recursively enumerable.*

To prove this extension of Corollary 5.5 it suffices to show that in $\varphi_{\mathcal{T}} \wedge \varphi_C$

1. all non-monadic predicates can be replaced by formulas containing monadic predicates only,
2. all predicates can be replaced by formulas containing only constant symbols and the equality symbol

in such a way that the resulting formulas are satisfiable in a first order Kripke structure validating $\mathbf{S5.C}^Q$ whenever there exists a recurrent tiling of $\omega \times \omega$ by $\langle \mathcal{T}, T_0 \rangle$.

Denote by $\varphi_{\mathcal{T}}^m$ and φ_C^m the formulas obtained from $\varphi_{\mathcal{T}}$ and φ_C when every occurence of $R(x,y)$ and $R_0(x,y)$ (x, y arbitrary individual variables) is replaced by

$$\neg\mathbf{K}_1\neg(Q_1(x) \wedge Q_2(y)) \text{ and } \neg\mathbf{K}_1\neg(Q_3(x) \wedge Q_4(y)),$$

respectively. Here $Q_1, \ldots, Q_4$ are unary predicates.

Denote by $\varphi_{\mathcal{T}}^c$ and φ_C^c the formulas obtained from $\varphi_{\mathcal{T}}$ and φ_C when every occurence of $R(x,y)$ and $R_0(x,y)$ is replaced by

$$\neg\mathbf{K}_1\neg(x = c_1 \wedge y = c_2) \text{ and } \neg\mathbf{K}_1\neg(x = c_3 \wedge y = c_4),$$

and every occurence of $P_i(x), i \leq M$, is replaced by

$$q_i \wedge \neg\mathbf{K}_1\neg(q_i \wedge x = d_i).$$

Here $c_1, \ldots, c_4$ and $d_i, i \leq M$, are local constant symbols and q_i, $i \leq M$, are 0-ary predicate symbols. We use the predicate symbols q_i and p for simplicity only: of course they can be replaced by equations $e_i = e$, for constant symbols e and $e_i, i \leq M$. So we may assume that $\varphi_{\mathcal{T}}^c$ and φ_C^c contain constants and the equality symbol only.

LEMMA 5.7 *If $f : \omega \times \omega \to \mathcal{T}$ is a tiling such that $\{n \in \omega : f(0,n) = T_0\}$ is infinite, then both $\varphi_{\mathcal{T}}^m \wedge \varphi_C^m$ and $\varphi_{\mathcal{T}}^c \wedge \varphi_C^c$ are satisfiable in a first order Kripke structure validating $\mathbf{S5}.\mathbf{C}^Q$.*

Proof Suppose $f : \omega \times \omega \to \mathcal{T}$ is a tiling such that $\{n \in \omega : f(0,n) = T_0\}$ is infinite. Take the first order Kripke structure $\langle \mathcal{F}, D, S \rangle$ defined in the proof of Theorem 5.3. The required structure $\langle \mathcal{F}', D', S' \rangle$ will be obtained by introducing infinitely many copies of each $S(n)$, $n \in \omega$. More precisely, define $\mathcal{F}' = \langle W', T_1, T_2 \rangle$ by putting

$$W' = (\omega \times A) \cup \mathsf{succ},$$

for a countably infinite set A disjoint from ω. Let T_1, T_2 be the reflexive, transitive, and symmetrical closures of T_1' and T_2', where

$$T_1' = \{((n,m),x) : n \in \omega, m \in A, x \in \mathsf{succ}, nS_1x\}$$

and

$$\begin{aligned} T_2' \;=\; & \{(x,(n,m)) : n \in \omega, m \in A, x \in \mathsf{succ}, xS_2n\} \cup \\ & \{((n,m_1),(n,m_2)) : n \in \omega, m_1, m_2 \in A\}. \end{aligned}$$

Consider, for $n \in \omega$, surjective functions

$$(f_n^1, f_n^2) : A \to R^{S(n)} \text{ and } (g_n^1, g_n^2) : A \to R_0^{S(n)}$$

such that $(t_n^1, t_n^2)(m) = (t_n^1(m), t_n^2(m))$, for $t \in \{f, g\}$. Take, for $n \in \omega$, and $i \leq M$ with $P_i^{S(n)} \neq \emptyset$ surjective functions

$$t_n^i : A \to P_i^{S(n)}.$$

Now put $D' = D$ and define S' as follows: for $(n, m) \in \omega \times A$ put

$$\begin{aligned} S'(n,m) &= \langle D, p^{S'(n,m)}, q_0^{S'(n,m)}, \ldots, R^{S'(n,m)}, R_0^{S'(n,m)}, P_0^{S'(n,m)}, \ldots, \\ &\quad Q_0^{S'(n,m)}, \ldots, c_0^{S'(n,m)}, \ldots, d_0^{S'(n,m)}, \ldots \rangle, \end{aligned}$$

where

- $p^{S'(n,m)} = p^{S(n)}$, $R^{S'(n,m)} = R^{S(n)}$, $R_0^{S'(n,m)} = R_0^{S(n)}$.
- $P_i^{S'(n,m)} = P_i^{S(n)}$, for all $i \leq M$.
- $q_i^{S'(n,m)} = 1$ iff $P_i^{S(n)} \neq \emptyset$, for all $i \leq M$.
- $c_0^{S'(n,m)} = f_n^1(m)$, $c_1^{S'(n,m)} = f_n^2(m)$.
- $c_3^{S'(n,m)} = g_n^1(m)$, $c_4^{S'(n,m)} = g_n^2(m)$.
- $Q_j^{S'(n,m)} = \{c_j^{S'(n,m)}\}$, for $1 \leq j \leq 4$.
- $d_i^{S'(n,m)} = h_n^i(m)$ whenever $P_i^{S(n)} \neq \emptyset$, for all $i \leq M$. Otherwise let $d_i^{S'(n,m)} \in D'$ arbitrary.

For $(n, n+1) \in \mathsf{succ}$ put

$$\begin{aligned} S'(n,n+1) &= \langle D, p^{S'(n,n+1)}, q_0^{S'(n,n+1)}, \ldots R^{S'(n,n+1)}, R_0^{S'(n,n+1)}, \\ &\quad P_0^{S'(n,n+1)}, \ldots, Q_0^{S'(n,n+1)}, \ldots, c_0^{S'(n,n+1)}, \ldots, d_0^{S'(n,n+1)}, \ldots \rangle \end{aligned}$$

where

- $p^{S'(n,n+1)} = p^{S(n,n+1)} = 0$.
- $R^{S'(n,n+1)} = R^{S(n,n+1)}$, $R_0^{S'(n,n+1)} = R_0^{S(n,n+1)}$.
- $P_i^{S'(n,n+1)} = P_i^{S(n,n+1)} = \emptyset$, for all $i \leq M$.
- both $c_0^{S'(n,n+1)}$ and $c_1^{S'(n,n+1)}$ are chosen arbitrarily in such a way that $(c_0^{S'(n,n+1)}, c_1^{S'(n,n+1)}) \in R^{S'(n,n+1)}$.
- both $c_3^{S'(n,n+1)}$ and $c_4^{S'(n,n+1)}$ are chosen arbitrarily in such a way that $(c_3^{S'(n,n+1)}, c_4^{S'(n,n+1)}) \in R_0^{S'(n,n+1)}$.

- $Q_j^{S'(n,n+1)} = \{c_j^{S'(n,n+1)}\}$, for all $j \leq M$.
- $q_i^{S'(n,n+1)} = 0$, for all $i \leq M$.
- $d_i^{S'(n,n+1)}$ is chosen arbitrarily, for all $i \leq M$.

Now it is readily checked that, for all $w \in W'$ and all $x, y \in D'$:

- $w \models R(x,y)$ iff $w \models \neg\mathbf{K}_1\neg(Q_1(x) \wedge Q_2(y))$ iff $w \models \neg\mathbf{K}_1\neg(x = c_1 \wedge x = c_2)$.
- $w \models R_0(x,y)$ iff $w \models \neg\mathbf{K}_1\neg(Q_3(x) \wedge Q_4(y))$ iff $w \models \neg\mathbf{K}_1\neg(x = c_3 \wedge x = c_4)$.
- $w \models P_i(x)$ iff $w \models (q_i \wedge \neg\mathbf{K}_1\neg(q_i \wedge x = d_i))$, for all $i \leq M$.

Hence both $\varphi_{\mathcal{T}}^m \wedge \varphi_C^m$ and $\varphi_{\mathcal{T}}^c \wedge \varphi_C^c$ are satisfied in $\langle \mathcal{F}', D', S' \rangle$. ❑

Theorem 5.6 follows immediately.

So reasoning in first order common knowledge logics is highly undecidable even for rather weak fragments. The question arises wether there are interesting decidable fragments of (standard) first order common knowledge logics at all. This question was answered in the positive in [15] for the following fragment $\mathcal{CL}$: let $\varphi \in \mathcal{CEL}^=$ be a formula containing local predicate symbols and global constants only. Then $\varphi \in \mathcal{CL}$ iff φ contains no subformula of the form $\Box\psi$, $\Box \in \{\mathbf{C}, \mathbf{K}_1, \mathbf{K}_2\}$, such that ψ contains more than one free variable. $\mathcal{CL}$ is undecidable since its first order non-epistemic fragment is undecidable. However, using the technique developed in [15] one can prove:

THEOREM 5.8 *Let L be a standard first order common knowledge logic.*

- *Let $\Sigma \subseteq \mathcal{CL}$ be the fragment of $\mathcal{CL}$ containing formulas with monadic predicates and constants only. Then $\Sigma \cap L$ is decidable.*
- *Let $\Sigma \subseteq \mathcal{CL}$ be the fragment of $\mathcal{CL}$ containing formulas with two variables only. Then $\Sigma \cap L$ is decidable.*

Those results actually follow from a rather general criterion for the decidability of fragments of modal predicate logics. We refer the reader to [15] for more information about fragments of this type.

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