More on Interpolants and Explicit Definitions for Description Logics with Nominals and/or Role Inclusions

Jean Christoph Jung[1], Andrea Mazzullo[2] and Frank Wolter[3]

[1] *University of Hildesheim*
[2] *Free University of Bozen-Bolzano*
[3] *University of Liverpool*

Abstract

It is known that the problems of deciding the existence of Craig interpolants and of explicit definitions of concepts are both 2ExpTime-complete for standard description logics with nominals and/or role inclusions. These complexity results depend on the presence of an ontology. In this article, we first consider the case without ontologies (or, in the case of role inclusions, ontologies only containing role inclusions) and show that both the existence of Craig interpolants and of explicit definitions of concepts become coNExpTime-complete for DLs such as $\mathcal{ALCO}$ and $\mathcal{ALCH}$. Secondly, we make a few observations regarding the size and computation of interpolants and explicit definitions.

Keywords

Craig interpolants, Explicit definitions, Beth definability property, Description logics

1. Introduction

Craig interpolants and explicit definitions have many potential applications in ontology engineering and ontology-based information systems. Examples include the extraction of equivalent acyclic TBoxes from ontologies [1, 2], the computation of referring expressions (or definite descriptions) for individuals [3], the equivalent rewriting of ontology-mediated queries into concepts or formulas [4, 5, 6, 7, 8], the construction of alignments between ontologies [9], and the decomposition of ontologies [10]. For logics enjoying the Craig interpolation property (CIP) the existence of a Craig interpolant follows from the validity of the defining subsumption and for logics enjoying the projective Beth definability property (PBDP) the existence of an explicit definition of a concept follows from its implicit definability. For such logics, deciding the existence of a Craig interpolant or an explicit definition of a concept are therefore not harder than subsumption and can be decided in ExpTime for DLs such as $\mathcal{ALC}$, $\mathcal{ALCI}$, $\mathcal{ALCQI}$ (which enjoy the CIP/PBDP [2]) if an ontology is present, and in PSpace without ontology.

This paper is a part of a research programme with the goal of understanding Craig interpolants and explicit definitions for logics that do not enjoy the CIP/PBDP [11, 12]. The two most basic constructors that lead to DLs without the CIP and PBDP are nominals and role inclusions. In

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jungj@uni-hildesheim.de (J. C. Jung); mazzullo@inf.unibz.it (A. Mazzullo); wolter@liverpool.ac.uk (F. Wolter)

fact, in [13], it is shown that the complexity of deciding the existence of Craig interpolants and explicit definitions are both 2ExpTime complete for standard DLs containing $\mathcal{ALCO}$ or $\mathcal{ALCH}$ and contained in the extension of $\mathcal{ALCHIO}$ with the universal role, in the presence of an ontology. The case without ontology remained open. Note that nothing interesting happens for DLs containing the universal role or both nominals and inverse roles as it is known that then the ontology can be 'internalised', and thus there is no difference between the case with and without ontology. For DLs such as $\mathcal{ALCO}$, $\mathcal{ALCH}$, and $\mathcal{ALCHI}$, however, this is not the case. In fact, it is known that subsumption checking becomes PSpace-complete without ontology while it is ExpTime-complete with ontology. In the first part of this paper we investigate the complexity of deciding the existence of Craig interpolants and explicit definitions without ontologies for these DLs and show that it becomes coNExpTime-complete. Hence we observe again a significant increase in complexity compared to subsumption checking. Note that for $\mathcal{ALCH}$ we assume an ontology containing role inclusions only as otherwise they cannot be introduced and are not relevant.

In practice, of course, one is interested in the actual interpolants or the explicit definition. Unfortunately, the decision procedures for the existence problems provided in this paper and in [13] are non-constructive in the sense that they do not return an interpolant (an explicit definition) in case it exists. To address this problem, we (slightly) modify the decision procedure from [13] and show how to read off interpolants / explicit definitions from a successful run, at least for DLs with role inclusions. In doing so, we take inspiration from a recent note on a type elimination based computation of interpolants in modal logic [14] which was originally provided for the guarded fragment [15].

For a discusson of further related work on interpolation, Beth definability, interpolant existence, and explicit definition existence we refer the reader to [11, 13].

2. Preliminaries

We first introduce standard DL definitions and notation [16]. Let $\mathsf{N_C}$, $\mathsf{N_R}$, and $\mathsf{N_I}$ be mutually disjoint and countably infinite sets of *concept*, *role*, and *individual names*. A *role* is a role name s or an *inverse role* s^-, with s a role name and $(s^-)^- = s$. We use u to denote the *universal role*. A *nominal* takes the form $\{a\}$, with $a \in \mathsf{N_I}$. An $\mathcal{ALCIO}^u$*-concept* is defined by the syntax rule

$$C, D ::= \top \mid A \mid \{a\} \mid \neg C \mid C \sqcap D \mid \exists r.C$$

where $a \in \mathsf{N_I}$, $A \in \mathsf{N_C}$, and r is a role. We use $C \sqcup D$ as abbreviation for $\neg(\neg C \sqcap \neg D)$, $C \rightarrow D$ for $\neg C \sqcup D$, and $\forall r.C$ for $\neg \exists r.(\neg C)$. We also consider the following fragments of $\mathcal{ALCIO}^u$: $\mathcal{ALCIO}$, obtained by dropping the universal role; $\mathcal{ALCO}^u$, obtained by dropping inverse roles; $\mathcal{ALCO}$, obtained from $\mathcal{ALCO}^u$ by dropping the universal role; and $\mathcal{ALC}$, obtained from $\mathcal{ALCO}$ by dropping nominals. If $\mathcal{L}$ is any of the DLs defined above, then an $\mathcal{L}$*-concept inclusion ($\mathcal{L}$-CI)* takes the form $C \sqsubseteq D$, with C and D $\mathcal{L}$-concepts. An $\mathcal{L}$*-ontology* is a finite set of $\mathcal{L}$-CIs. We also consider DLs with *role inclusions (RIs)*, expressions of the form $r \sqsubseteq s$, where r and s are roles. As usual, the addition of RIs is indicated by adding the letter $\mathcal{H}$ to the name of the DL, where inverse roles occur in RIs only if the DL admits inverse roles. Thus, for example, $\mathcal{ALCH}$*-ontologies* are finite sets of $\mathcal{ALC}$-CIs and RIs not using inverse roles and

[AtomC]	for all $(d, e) \in S$: $d \in A^{\mathcal{I}}$ iff $e \in A^{\mathcal{J}}$
[AtomI]	for all $(d, e) \in S$: $d = a^{\mathcal{I}}$ iff $e = a^{\mathcal{J}}$
[Forth]	if $(d, e) \in S$ and $(d, d') \in r^{\mathcal{I}}$, then there is e' with $(e, e') \in r^{\mathcal{J}}$ and $(d', e') \in S$.
[Back]	if $(d, e) \in S$ and $(e, e') \in r^{\mathcal{J}}$, then there is d' with $(d, d') \in r^{\mathcal{I}}$ and $(d', e') \in S$.

Table 1
Conditions on $S \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{J}}$.

$\mathcal{ALCHIO}^u$*-ontologies* are finite sets of $\mathcal{ALCIO}^u$-CIs and RIs. In the following, we use $\mathsf{DL}_{\mathsf{nr}}$ to denote the set of DLs $\mathcal{ALCO}$, $\mathcal{ALCIO}$, $\mathcal{ALCH}$, $\mathcal{ALCHO}$, $\mathcal{ALCHIO}$, and their extensions with the universal role. To simplify notation we do not drop the letter $\mathcal{H}$ when speaking about the concepts and CIs of a DL with RIs. Thus, for example, we sometimes use the expressions $\mathcal{ALCHO}$-concept and $\mathcal{ALCHO}$-CI to denote $\mathcal{ALCO}$-concepts and CIs, respectively.

The semantics is given in terms of *interpretations* $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, defined as usual [16]. An interpretation $\mathcal{I}$ *satisfies* an $\mathcal{L}$-CI $C \sqsubseteq D$ if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ and an RI $r \sqsubseteq s$ if $r^{\mathcal{I}} \subseteq s^{\mathcal{I}}$. We say that $\mathcal{I}$ is a *model* of an ontology $\mathcal{O}$ if it satisfies all inclusions in it. We say that an inclusion α follows from an ontology $\mathcal{O}$, in symbols $\mathcal{O} \models \alpha$, if every model of $\mathcal{O}$ satisfies α. We write $\mathcal{O} \models C \equiv D$ if $\mathcal{O} \models C \sqsubseteq D$ and $\mathcal{O} \models D \sqsubseteq C$. We write $\models C \sqsubseteq D$ if $\mathcal{O} \models C \sqsubseteq D$ for the empty ontology $\mathcal{O}$. A concept C is *satisfiable* w.r.t. an ontology $\mathcal{O}$ if there is a model $\mathcal{I}$ of $\mathcal{O}$ with $C^{\mathcal{I}} \neq \emptyset$.

A *signature* Σ is a set of *symbols*, i.e., concept, role, and individual names. As standard in the literature, the universal role is not regarded as a symbol, but as a logical connective, and as such it is not contained in any signature. We use $\text{sig}(X)$ to denote the set of symbols used in any syntactic object X such as a concept or an ontology. An $\mathcal{L}(\Sigma)$*-concept* is an $\mathcal{L}$-concept C with $\text{sig}(C) \subseteq \Sigma$, and a Σ*-role* is a role r such that r or r^- is in Σ.

We next require preliminary model-theoretic definitions and results for the DLs $\mathcal{L}$ introduced above. First, we call *pointed interpretation* a pair $\mathcal{I}, d$ with $\mathcal{I}$ an interpretation and $d \in \Delta^{\mathcal{I}}$. For pointed interpretations $\mathcal{I}, d$ and $\mathcal{J}, e$ and a signature Σ, we write $\mathcal{I}, d \equiv_{\mathcal{L},\Sigma} \mathcal{J}, e$ and say that $\mathcal{I}, d$ and $\mathcal{J}, e$ are $\mathcal{L}(\Sigma)$*-equivalent* if $d \in C^{\mathcal{I}}$ iff $e \in C^{\mathcal{J}}$, for all $\mathcal{L}(\Sigma)$-concepts C.

Then, for a signature Σ, we call a relation $S \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{J}}$ an $\mathcal{ALC}(\Sigma)$*-bisimulation* if conditions [AtomC], [Forth] and [Back] from Table 1 hold, where A and r range, respectively, over all concept and role names in Σ. We write $\mathcal{I}, d \sim_{\mathcal{ALC},\Sigma} \mathcal{J}, e$ and call $\mathcal{I}, d$ and $\mathcal{J}, e$ $\mathcal{ALC}(\Sigma)$-*bisimilar* if there exists an $\mathcal{ALC}(\Sigma)$-bisimulation S such that $(d, e) \in S$. For $\mathcal{ALCO}$, we define $\sim_{\mathcal{ALCO},\Sigma}$ analogously, with the additional requirement that also condition [AtomI] in Table 1 holds, for all individual names $a \in \Sigma$. For a DL $\mathcal{L}$ with inverse roles, we require that, in Table 1, r additionally ranges over inverse roles. For a DL $\mathcal{L}$ with the universal role, we extend the respective conditions for the fragment $\mathcal{L}'$ of $\mathcal{L}$ without the universal role by demanding that the domain $\text{dom}(S)$ and range $\text{ran}(S)$ of S contain $\Delta^{\mathcal{I}}$ and $\Delta^{\mathcal{J}}$, respectively. If a DL $\mathcal{L}$ has RIs, then we use $\mathcal{I}, d \sim_{\mathcal{L},\Sigma} \mathcal{J}, e$ to state that $\mathcal{I}, d \sim_{\mathcal{L}',\Sigma} \mathcal{J}, e$ for the fragment $\mathcal{L}'$ of $\mathcal{L}$ without RIs.

Finally, the following lemma recalls the model-theoretic characterisations for objects in interpretations being indistinguishable by concepts expressed in one of the DLs introduced above [17, 18]. We refer the reader to [19] for the definition of ω-saturated structures.

Lemma 1. *Let $\mathcal{I}, d$ and $\mathcal{J}, e$ be pointed interpretations and ω-saturated. Let $\mathcal{L} \in \mathsf{DL}_{\mathsf{nr}}$ and Σ a* *signature. Then*

$$\mathcal{I}, d \equiv_{\mathcal{L},\Sigma} \mathcal{J}, e \quad \textit{iff} \quad \mathcal{I}, d \sim_{\mathcal{L},\Sigma} \mathcal{J}, e.$$

For the "if"-direction, the ω-saturatedness condition can be dropped.

3. Basic notions and results

Let $\mathcal{L}$ be a DL, let $\mathcal{O}_1, \mathcal{O}_2$ be $\mathcal{L}$-ontologies, and let C_1, C_2 be $\mathcal{L}$-concepts. We set $\text{sig}(\mathcal{O}, C) = \text{sig}(\mathcal{O}) \cup \text{sig}(C)$, for any ontology $\mathcal{O}$ and concept C. Following [2], an $\mathcal{L}$-concept D is called an *$\mathcal{L}$-interpolant* for $C_1 \sqsubseteq C_2$ under $\mathcal{O}_1 \cup \mathcal{O}_2$ if

- $\text{sig}(D) \subseteq \text{sig}(\mathcal{O}_1, C_1) \cap \text{sig}(\mathcal{O}_2, C_2)$;
- $\mathcal{O}_1 \cup \mathcal{O}_2 \models C_1 \sqsubseteq D$;
- $\mathcal{O}_1 \cup \mathcal{O}_2 \models D \sqsubseteq C_2$.

$\mathcal{L}$-interpolant existence is the problem to decide the existence of an interpolant for $C_1 \sqsubseteq C_2$ under $\mathcal{O}_1 \cup \mathcal{O}_2$. In logics with the appropriate Craig Interpolation Property (CIP) (such as, for instance, $\mathcal{ALC}$ and $\mathcal{ALCI}$ [2]) the existence of an $\mathcal{L}$-interpolant for $C_1 \sqsubseteq C_2$ under $\mathcal{O}_1 \cup \mathcal{O}_2$ is equivalent to the entailment $\mathcal{O}_1 \cup \mathcal{O}_2 \models C_1 \sqsubseteq C_2$ and thus reduces to standard subsumption checking (which is, for instance, ExpTime-complete for $\mathcal{ALC}$ and $\mathcal{ALCI}$). This is not the case for the DLs considered here; in fact the following increase in complexity by one exponential is shown in [13].

Theorem 2. *Let $\mathcal{L} \in \mathsf{DL}_{\mathsf{nr}}$. Then $\mathcal{L}$-interpolant existence is* 2ExpTime*-complete.*

In this article we consider interpolant existence with either empty ontologies or ontologies containing RIs only. In detail, *ontology-free $\mathcal{L}$-interpolant existence* is the problem to decide $\mathcal{L}$-interpolant existence for empty ontologies. Note that for logics with the CIP ontology-free interpolant existence reduces to checking $\models C_1 \sqsubseteq C_2$ and hence is PSpace-complete for DLs such as $\mathcal{ALC}$ and $\mathcal{ALCI}$. If $\mathcal{L}$ admits RIs, then we consider *ontology-free $\mathcal{L}$-interpolant existence with RIs*, the problem to decide $\mathcal{L}$-interpolant existence for ontologies containing RIs only. We observe that DLs in $\mathsf{DL}_{\mathsf{nr}}$ do not enjoy the CIP, even without ontologies (ontologies containing RIs only, respectively).

Example 1. *Consider $C_1 = \{a\} \sqcap \exists r.\{a\}$ and $C_2 = \{b\} \rightarrow \exists r.\{b\}$. Then $\models C_1 \sqsubseteq C_2$ but there does not exist any $\mathcal{ALCO}$-interpolant for $C_1 \sqsubseteq C_2$ (see Example 5 for a proof). An example using RIs instead of nominals can be constructed from Example 3 below.*

We next introduce explicit definitions. We call an $\mathcal{L}(\Sigma)$-concept D an *explicit $\mathcal{L}(\Sigma)$-definition of C_0 under an ontology $\mathcal{O}$* if $\mathcal{O} \models C_0 \equiv D$. *$\mathcal{L}$-explicit definition* is the problem to decide the existence of an $\mathcal{L}(\Sigma)$-definition of an $\mathcal{L}$-concept under an $\mathcal{L}$-ontology. In logics with the appropriate projective Beth Definability Property (PBDP) [2, 13] the existence of an explicit $\mathcal{L}(\Sigma)$-definition of a concept follows from its implicit definability which can be decided using subsumption checking and is therefore ExpTime-complete for DLs with the PBDP such as $\mathcal{ALC}$ and $\mathcal{ALCI}$ [2]. Similarly to the interpolant existence problem, this is not the case for the DLs considered here and we have again an increase in complexity by one exponential [13].

Theorem 3. *Let $\mathcal{L} \in \mathsf{DL}_{\mathsf{nr}}$. Then explicit definition existence is* 2ExpTime*-complete.*

In this article we consider explicit definition existence without ontologies and ontologies containing RIs only. If C and C_0 are concepts and Σ a signature, then we call D an *explicit $\mathcal{L}(\Sigma)$-definition of C_0 under C* if $\models C \rightarrow (C_0 \leftrightarrow D)$.

Example 2. *Explicit definitions under a concept C can be regarded as a 'local' version of explicit definitions under ontologies. If $\mathcal{O}$ is an ontology, then let N_n be the concept stating that $\mathcal{O}$ is true in all nodes reachable in at most n steps. Then a concept D is an $\mathcal{L}(\Sigma)$-definition of C_0 under $\mathcal{O}$ iff there exists an $n \geq 0$ such that D is an $\mathcal{L}(\Sigma)$-definition of C_0 under N_n.*

Then *ontology-free $\mathcal{L}$-definition existence* is the problem to decide for $\mathcal{L}$-concepts C and C_0, and a signature Σ whether there exists an explicit $\mathcal{L}(\Sigma)$-definition of C_0 under C. If $\mathcal{L}$ admits RIs, then *ontology-free $\mathcal{L}$-definition existence with RIs* is the problem to decide for an ontology $\mathcal{O}$ containing RIs only, $\mathcal{L}$-concepts C and C_0, and a signature Σ whether there exists an explicit $\mathcal{L}(\Sigma)$-definition of C_0 under $\mathcal{O}$ and C, that is $\mathcal{O} \models C \rightarrow (C_0 \leftrightarrow D)$. For DLs with the PBDP such as $\mathcal{ALC}$ and $\mathcal{ALCI}$ ontology-free $\mathcal{L}$-definition existence reduces to subsumption checking without ontologies and is thus PSpace-complete. We next observe that the DLs in $\mathsf{DL}_{\mathsf{nr}}$ do not enjoy the PBDP without ontologies (ontologies containing RIs only).

Example 3. *Consider $\mathcal{O} = \{r \sqsubseteq r_1, r \sqsubseteq r_2\}$ and let C be the conjunction of $C_1 = (\neg\exists r.\top \sqcap \exists r_1.A) \rightarrow \forall r_2.\neg A$ and $C_2 = (\neg\exists r.\top \sqcap \exists r_1.\neg A) \rightarrow \forall r_2.A$. Let $\Sigma = \{r_1, r_2\}$. Then there does not exist an explicit $\mathcal{ALC}(\Sigma)$-definition of $\exists r.\top$ under $\mathcal{O}$ and C (see Example 6 below for a proof). The concept $\exists r_1 \cap r_2.\top$, however, is an explicit definition of $\exists r.\top$ under $\mathcal{O}$ and C in the extension of $\mathcal{ALC}$ with role intersection (with semantics defined in the obvious way). Hence $\exists r.\top$ is implicitly definable. A counterexample to the ontology-free PBDP for DLs with nominals is constructed in the lower bound proof below.*

We conclude this section with a few observations on the relationship between the existence problems introduced above. In particular, by applying a standard encoding of ontologies into concepts we show that for DLs in $\mathsf{DL}_{\mathsf{nr}}$ containing the universal role or both inverse roles and nominals dropping the ontology does not affect their complexity.

It has been observed in [13] already that $\mathcal{L}$-explicit definition existence is polyomial time reducible to $\mathcal{L}$-interpolant existence. This also holds for the ontology-free versions.

Lemma 4. *Let $\mathcal{L} \in \mathsf{DL}_{\mathsf{nr}}$. Then ontology-free $\mathcal{L}$-definition existence (with RIs) can be reduced in polynomial time to ontology-free $\mathcal{L}$-interpolant existence (with RIs).*

Proof. Assume $\mathcal{O}, C, C_0$, and Σ are given. Then there exists an explicit $\mathcal{L}(\Sigma)$-definition of C_0 under $\mathcal{O}$ and C iff there exists an $\mathcal{L}$-interpolant for $C \sqcap C_0 \sqsubseteq C_\Sigma \rightarrow C_{0\Sigma}$ under $\mathcal{O}, \mathcal{O}_\Sigma$, where $\mathcal{O}_\Sigma$, $C_{0\Sigma}$, and C_Σ are obtained from $\mathcal{O}$, C_0, and C by replacing every symbol not in Σ by a fresh symbol. ❏

Lemma 5. *Let $\mathcal{L} \in \mathsf{DL_{nr}}$ contain the universal role or both inverse roles and nominals. Then $\mathcal{L}$-explicit definition existence can be reduced in polynomial time to ontology-free $\mathcal{L}$-definition existence (with RIs if $\mathcal{L}$ admits RIs).*

Proof. Assume $\mathcal{O}$, C_0, and Σ are given. We may assume that $\mathcal{O}$ takes the form $\{\top \sqsubseteq D\} \cup \mathcal{O}'$ with $\mathcal{O}'$ a set of RIs.

Assume first that $\mathcal{L}$ admits the universal role. Then one can easily show that there exists an explicit $\mathcal{L}(\Sigma)$-definition of C_0 under $\mathcal{O}$ iff there exists an explicit $\mathcal{L}(\Sigma)$-definition of C_0 under $\mathcal{O}'$ and $\forall u.D$.

Now assume that $\mathcal{L}$ admits inverse roles and nominals. We use the spy-point technique to encode the universal role. Introduce a fresh individual a and a fresh role name r_0 and define U as the conjunction of the concepts $\{a\}$, $\exists r_0.\{a\}$, $\forall s.\exists r_0.\{a\}$, and $\forall r_0^-.\forall s.\exists r_0.\{a\}$, for all $s \in \{r, r^-\}$ such that r occurs in $\mathcal{O}$ or C_0. Observe that if $d \in (U \sqcap \forall r_0^-.C)^\mathcal{I}$ for some interpretations $\mathcal{I}$ then $e \in C^\mathcal{I}$ holds in all nodes e in $\Delta^\mathcal{I}$ that can be reached along roles occuring in $\mathcal{O}$ or C_0 from d. It follows that for any $\mathcal{L}(\Sigma)$-concept E, we have

$$\mathcal{O} \models C_0 \equiv E \quad \text{iff} \quad \mathcal{O}' \models (U \sqcap \forall r_0^-.D) \rightarrow (C_0 \leftrightarrow E)$$

Hence there exists an explicit $\mathcal{L}(\Sigma)$-definition of C_0 under $\mathcal{O}$ iff there exists an explicit $\mathcal{L}(\Sigma)$-definition of C_0 under $\mathcal{O}'$ and $U \sqcap \forall r_0^-.D$. ❏

We obtain the following complexity result as a consequence of Theorems 2 and 3 and Lemmas 4 and 5.

Theorem 6. *Let $\mathcal{L} \in \mathsf{DL_{nr}}$ contain the universal role or both inverse roles and nominals. Then ontology-free interpolant existence (with RIs) and ontology-free explicit definition existence (with RIs) are both* 2ExpTime*-complete.*

4. Joint Consistency

The first main concern of the present paper is to study the computational complexity of the ontology-free interpolant and explicit definition existence problems. As a preliminary step, we provide model-theoretic characterizations in terms of bisimulations as captured in the following central notion. Using bisimulations, we show the upper bound for a generalization of interpolant existence. *Generalized $\mathcal{L}$-interpolant existence* is the problem to decide for an $\mathcal{L}$ ontology $\mathcal{O}$, $\mathcal{L}$-concepts C_1, C_2 and signature Σ whether there exists an $\mathcal{L}(\Sigma)$*-interpolant* for $C_1 \sqsubseteq C_2$ under $\mathcal{O}$, that is, an $\mathcal{L}(\Sigma)$-concept D such that $\mathcal{O} \models C_1 \sqsubseteq D$ and $\mathcal{O} \models D \sqsubseteq C_2$. The ontology-free version and the version with ontologies containing RIs only are defined in the obvious way. Note that $\mathcal{L}$-interpolant existence is indeed a special case of generalized $\mathcal{L}$-interpolant existence by setting $\mathcal{O} = \mathcal{O}_1 = \mathcal{O}_2$ and $\Sigma = \text{sig}(\mathcal{O}_1, C_1) \cap \text{sig}(\mathcal{O}_2, C_2)$.

Definition 4 (Joint consistency). *Let $\mathcal{L} \in \mathsf{DL_{nr}}$. Let $\mathcal{O}$ be an $\mathcal{L}$-ontology, C_1, C_2 be $\mathcal{L}$-concepts, and $\Sigma \subseteq sig(\mathcal{O}, C_1, C_2)$ be a signature. Then C_1, C_2 are called* jointly consistent under $\mathcal{O}$ modulo $\mathcal{L}(\Sigma)$-bisimulations *if there exist pointed models $\mathcal{I}_1, d_1$ and $\mathcal{I}_2, d_2$ such that $\mathcal{I}_i$ is a model of $\mathcal{O}$, $d_i \in C_i^{\mathcal{I}_i}$, for $i = 1, 2$, and $\mathcal{I}_1, d_1 \sim_{\mathcal{L},\Sigma} \mathcal{I}_2, d_2$.*

The associated decision problem, *joint consistency modulo $\mathcal{L}$-bisimulations*, is defined in the expected way. The following result characterizes the existence of interpolants using joint consistency modulo $\mathcal{L}(\Sigma)$-bisimulations and is proved in [13].

Theorem 7. *Let $\mathcal{L} \in \mathsf{DL_{nr}}$. Let $\mathcal{O}$ be an $\mathcal{L}$-ontology, C_1, C_2 be $\mathcal{L}$-concepts, and $\Sigma \subseteq \mathrm{sig}(\mathcal{O}, C_1, C_2)$. Then the following conditions are equivalent:*

1. *there is no $\mathcal{L}(\Sigma)$-interpolant for $C_1 \sqsubseteq C_2$ under $\mathcal{O}$;*
2. *$C_1, \neg C_2$ are jointly consistent under $\mathcal{O}$ modulo $\mathcal{L}(\Sigma)$-bisimulations.*

Example 5. *Consider, from Example 1, $C_1 = \{a\} \sqcap \exists r.\{a\}$ and $C_2 = \{b\} \rightarrow \exists r.\{b\}$. The interpretations $\mathcal{I}_1, \mathcal{I}_2$ depicted in Figure 1 show that C_1 and $\neg C_2$ are jointly consistent modulo $\mathcal{ALCO}(\Sigma)$-bisimulations, where $\Sigma = \{r\}$.*

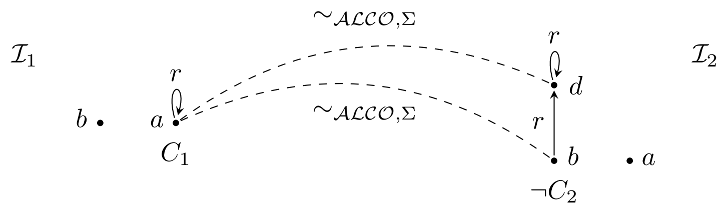

Figure 1: Interpretations $\mathcal{I}_1$ and $\mathcal{I}_2$ illustrating Example 5.

The following characterization of the existence of explicit definitions is a direct consequence of Theorem 7.

Theorem 8. *Let $\mathcal{L} \in \mathsf{DL_{nr}}$. Let $\mathcal{O}$ be an $\mathcal{L}$-ontology, C and C_0 $\mathcal{L}$-concepts, and $\Sigma \subseteq sig(\mathcal{O}, C)$ a signature. Then the following conditions are equivalent:*

1. *there is no explicit $\mathcal{L}(\Sigma)$-definition of C_0 under $\mathcal{O}$ and C;*
2. *$C \sqcap C_0$ and $C \sqcap \neg C_0$ are jointly consistent under $\mathcal{O}$ modulo $\mathcal{L}(\Sigma)$-bisimulations.*

Example 6. *Consider $\mathcal{O}$, C, and Σ from Example 3. The interpretations $\mathcal{I}_1, \mathcal{I}_2$ depicted in Figure 2 show that $C \sqcap \exists r.\top$ and $C \sqcap \neg\exists r.\top$ are jointly consistent under $\mathcal{O}$ modulo $\mathcal{ALCH}(\Sigma)$-bisimulations.*

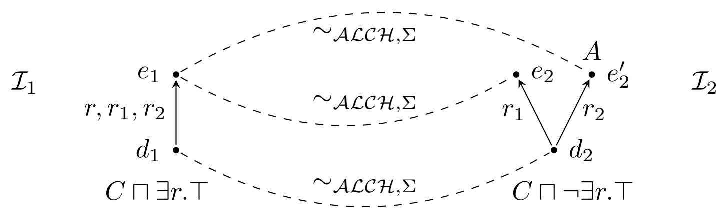

Figure 2: Interpretations $\mathcal{I}_1$ and $\mathcal{I}_2$ illustrating Example 6.

5. Complexity

We formulate our main complexity result about the problem of deciding the existence of interpolants and explicit definitions.

Theorem 9. *Let $\mathcal{L} \in \mathsf{DL}_{\mathsf{nr}}$ not contain the universal role and not contain both inverse roles and nominals simultaneously. Then ontology-free generalized $\mathcal{L}$-interpolant existence (with RIs), ontology-free $\mathcal{L}$-interpolant existence (with RIs), and ontology-free $\mathcal{L}$-definition existence (with RIs) are all* coNExpTime*-complete.*

We show the upper bound for generalized $\mathcal{L}$-interpolant existence by proving that joint consistency is in NExpTime (Theorem 7) and we show the lower bound by proving NExpTime-hardness for the version of joint consistency formulated in Theorem 8 (with empty ontology or, respectively, ontologies containing RIs only).

To show these results, we first require the following definitions. The *depth* of a concept C is the number of nestings of restrictions in C. For instance, a concept names has depth 0 and $\exists r \exists r.B$ has depth 2. Given an ontology $\mathcal{O}$ and concepts C_1, C_2, let $\Xi = \text{sub}(\mathcal{O}, C_1, C_2)$ denote the closure under single negation of the set of subconcepts of concepts in $\mathcal{O}, C_1, C_2$. A Ξ-*type* t is a subset of Ξ such that there exists an interpretation $\mathcal{I}$ and $d \in \Delta^{\mathcal{I}}$ with $t = \text{tp}_{\Xi}(\mathcal{I}, d)$, where

$$\text{tp}_{\Xi}(\mathcal{I}, d) = \{C \in \Xi \mid d \in C^{\mathcal{I}}\}$$

is the Ξ-*type realized at* d *in* $\mathcal{I}$. For a signature $\Sigma \subseteq \text{sig}(\mathcal{O}, C_1, C_2)$ and $i \in \{1, 2\}$, the *mosaic* defined by $d \in \Delta^{\mathcal{I}_i}$ in $\mathcal{I}_1, \mathcal{I}_2$ is the pair $(T_1(d), T_2(d))$ such that

$$T_j(d) = \{\text{tp}_{\Xi}(\mathcal{I}_j, e) \mid e \in \Delta^{\mathcal{I}_j}, \mathcal{I}_i, d \sim_{\mathcal{L},\Sigma} \mathcal{I}_j, e\},$$

for $j = 1, 2$. We say that a pair (T_1, T_2) of sets T_1, T_2 of types is a *mosaic defined by* $\mathcal{I}_1, \mathcal{I}_2$ if there exists $d \in \Delta^{\mathcal{I}_1} \cup \Delta^{\mathcal{I}_2}$ such that $(T_1, T_2) = (T_1(d), T_2(d))$.

Example 7. *From Example 5, consider C_1, C_2, as well as $\mathcal{I}_1, \mathcal{I}_2$. The set Ξ consists of the concepts $\{a\}$, $\exists r.\{a\}$, $\{b\}$, $\exists r.\{b\}$, C_1, C_2, and negations thereof. We have that:*

- $\text{tp}_{\Xi}(\mathcal{I}_1, a^{\mathcal{I}_1}) = \{\{a\}, \exists r.\{a\}, \neg\{b\}, \neg\exists r.\{b\}, C_1, C_2\}$;
- $\text{tp}_{\Xi}(\mathcal{I}_2, b^{\mathcal{I}_2}) = \{\neg\{a\}, \neg\exists r.\{a\}, \{b\}, \neg\exists r.\{b\}, \neg C_1, \neg C_2\}$;
- $\text{tp}_{\Xi}(\mathcal{I}_2, d) = \{\neg\{a\}, \neg\exists r.\{a\}, \neg\{b\}, \neg\exists r.\{b\}, \neg C_1, C_2\}$.

The mosaic defined by $a^{\mathcal{I}_1}$ in $\mathcal{I}_1, \mathcal{I}_2$ is $(T_1(a^{\mathcal{I}_1}), T_2(a^{\mathcal{I}_1}))$, where $T_1(a^{\mathcal{I}_1}) = \{\text{tp}_{\Xi}(\mathcal{I}_1, a^{\mathcal{I}_1})\}$ and $T_2(a^{\mathcal{I}_1}) = \{\text{tp}_{\Xi}(\mathcal{I}_2, b^{\mathcal{I}_2}), \text{tp}_{\Xi}(\mathcal{I}_2, d)\}$.

Example 8. *From Example 6, consider $\mathcal{O}$, C, Σ, as well as $\mathcal{I}_1, \mathcal{I}_2$, and let $D_1 = \neg\exists r.\top \sqcap \exists r_1.A$ and $D_2 = \neg\exists r.\top \sqcap \exists r_1.\neg A$. The set Ξ consists of the concepts $\top$, A, $\exists r.\top$, $\exists r_1.A$, $\exists r_1.\neg A$, $\forall r_2.A$, $\forall r_2.\neg A$, D_1, D_2, C_1, C_2, and negations thereof. We have that:*

- $\text{tp}_{\Xi}(\mathcal{I}_1, d_1) = \{\neg A, \exists r.\top, \exists r_1.\neg A, \neg\exists r_1.A, \forall r_2.\neg A, \neg\forall r_2.A, \neg D_1, \neg D_2, C_1, C_2, C\}$;
- $\text{tp}_{\Xi}(\mathcal{I}_1, e_1) = \{\neg A, \neg\exists r.\top, \neg\exists r_1.A, \neg\exists r_1.\neg A, \forall r_2.A, \forall r_2.\neg A, \neg D_1, \neg D_2, C_1, C_2, C\}$;

- $\text{tp}_\Xi(\mathcal{I}_2, d_2) = \{\neg A, \neg\exists r.\top, \exists r_1.\neg A, \neg\exists r_1.A, \forall r_2.A, \neg\forall r_2.\neg A, \neg D_1, D_2, C_1, C_2, C\}$;
- $\text{tp}_\Xi(\mathcal{I}_2, e_2) = \{\neg A, \neg\exists r.\top, \neg\exists r_1.A, \neg\exists r_1.\neg A, \forall r_2.A, \forall r_2.\neg A, \neg D_1, \neg D_2, C_1, C_2, C\}$.
- $\text{tp}_\Xi(\mathcal{I}_2, e_2') = \{A, \neg\exists r.\top, \neg\exists r_1.A, \neg\exists r_1.\neg A, \forall r_2.A, \forall r_2.\neg A, \neg D_1, \neg D_2, C_1, C_2, C\}$.

The mosaic defined by d_1 in $\mathcal{I}_1, \mathcal{I}_2$ is $(T_1(d_1), T_2(d_1))$, where $T_1(d_1) = \{\text{tp}_\Xi(\mathcal{I}_1, d_1)\}$ and $T_2(d_1) = \{\text{tp}_\Xi(\mathcal{I}_2, d_2)\}$. On the other hand, the mosaic defined by e_1 in $\mathcal{I}_1, \mathcal{I}_2$ is $(T_1(e_1), T_2(e_1))$, where $T_1(e_1) = \{\text{tp}_\Xi(\mathcal{I}_1, e_1)\}$ and $T_2(e_1) = \{\text{tp}_\Xi(\mathcal{I}_2, e_2), \text{tp}_\Xi(\mathcal{I}_2, e_2')\}$.

A mosaic is *nominal generated* if some type in it contains a nominals. Consider $p = (T_1(d), T_2(d))$ and $q = (T_1(d'), T_2(d'))$ such that there exists a role name $r \in \Sigma$ with $(d, d') \in r^{\mathcal{I}_i}$, for some $i \in \{1, 2\}$. Then define, for every role name s and $i \in \{1, 2\}$, relations $R^{s,i}_{p,q} \subseteq T_i(d) \times T_i(d')$ by setting $(t, t') \in R^{s,i}_{p,q}$ if there exist e, e' realizing t and t', respectively, with $(T_1(e), T_2(e)) = p$ and $(T_1(e'), T_2(e')) = q$, such that $(e, e') \in s^{\mathcal{I}_i}$.

The upper bound follows from the following exponential size model property result.

Lemma 10. *Let $\mathcal{L} \in \mathsf{DL}_{\mathsf{nr}}$ not contain the universal role and not contain both inverse roles and nominals simultaneously. Let $\mathcal{O}$ be a set of RIs , C_1, C_2 $\mathcal{L}$-concepts, and Σ a signature. If C_1 and $\neg C_2$ are jointly consistent under $\mathcal{O}$ modulo $\mathcal{L}(\Sigma)$-bisimulations, then there exist models of exponential size witnessing this; in more detail, there exist pointed models $\mathcal{I}, d$ and $\mathcal{J}, e$ of $\mathcal{O}$ of at most exponential size such that $d \in C_1^{\mathcal{I}}$, $e \notin C_2^{\mathcal{J}}$, and $\mathcal{I}, d \sim_{\mathcal{L},\Sigma} \mathcal{J}, e$.*

Proof. Assume that C_1 and $\neg C_2$ are jointly consistent under $\mathcal{O}$ modulo $\mathcal{L}(\Sigma)$-bisimulations. By definition, there exist pointed models $\mathcal{I}_1, d_1$ and $\mathcal{I}_2, d_2$ of $\mathcal{O}$ such that $d_1 \in C_1^{\mathcal{I}_1}$, $d_2 \notin C_2^{\mathcal{I}_2}$, and $\mathcal{I}_1, d_1 \sim_{\mathcal{L},\Sigma} \mathcal{I}_2, d_2$. Let k be the maximum depth of C_1, C_2.

We start with the case involving nominals and without inverse roles. We construct exponential size $\mathcal{J}_1, \mathcal{J}_2$ with the same properties of $\mathcal{I}_1, \mathcal{I}_2$ above. Let $\mathcal{B}$ be some minimal set of mosaics defined by $\mathcal{I}_1, \mathcal{I}_2$ such that

- all nominal generated mosaics are in $\mathcal{B}$;
- for every type t realized in $\mathcal{I}_i$ there exists $(T_1, T_2) \in \mathcal{B}$ with $t \in T_i$;
- $(T_1(d_1), T_2(d_1)) \in \mathcal{B}$.

Observe that the size of $\mathcal{B}$ is at most exponential in the size of $\mathcal{O}, C_1, C_2$. Now select, for any mosaic $p = (T_1, T_2)$ defined by $\mathcal{I}_1, \mathcal{I}_2$ and any $\exists s.C \in t \in T_i$ such that there exists $r \in \Sigma$ with $\mathcal{O} \models s \sqsubseteq r$, a mosaic $q = (T_1', T_2')$ such that $(t, t') \in R^{s,i}_{p,q}$ and $C \in t'$, and denote the resulting set by $\mathcal{S}(p)$. Form the set $\mathcal{T}$ of sequences

$$\sigma = p_0 \cdots p_j = (T_1^0, T_2^0) \cdots (T_1^j, T_2^j)$$

with $j \leq k$, $p_0 \in \mathcal{B}$ and $p_{i+1} \in \mathcal{S}(p_i)$ for $i < j$. Let $\text{tail}(\sigma) = p_j$ and $\text{tail}_i(\sigma) = T_i^j$. We next define the domain of $\mathcal{J}_1$ and $\mathcal{J}_2$ as

$$\begin{aligned} \Delta^{\mathcal{J}_i} \quad = \quad & \{(t, p) \mid t \in \text{tail}_i(p), p \in \mathcal{B}\} \cup \\ & \{(t, \sigma) \mid \sigma \in \mathcal{T}, t \in \text{tail}_i(\sigma), |\sigma| > 1, t \text{ contains no nominal}\} \end{aligned}$$

We define interpretations $\mathcal{J}_1, \mathcal{J}_2$ in the expected way.

- For any individual name a and $(T_1, T_2) \in \mathcal{B}$ with $\{a\} \in t \in T_i$, we set $a^{\mathcal{J}_i} = (t, (T_1, T_2))$.
- For any concept name A, $(t, \sigma) \in A^{\mathcal{J}_i}$ iff $A \in t$;
- Let r be a role name. Then we let for $\sigma p \in \mathcal{T}$,
  - $((t, \sigma), (t', \sigma p)) \in r^{\mathcal{J}_i}$ if $(t, t') \in R^{r,i}_{\mathsf{tail}(\sigma),p}$ and t' contains no nominal;
  - $((t, \sigma), (t', p)) \in r^{\mathcal{J}_i}$ if $(t, t') \in R^{r,i}_{\mathsf{tail}(\sigma),p}$ and t' contains a nominal.

  Next assume that $\mathsf{tail}(\sigma) = (T_1, T_2)$ and σ has length k. If $\mathsf{tail}(\sigma') = (T_1, T_2)$ for some $|\sigma'| < k$, then choose as r-successors of any node of the form (t, σ) exactly the r-successors of (t, σ') defined above. If no such σ' exists, then all nodes of the form $(t, \mathsf{tail}(\sigma))$ have distance exactly k from the roots (since no nominal occurs in any type in any mosaic in σ) and no successors are added.

  It remains to take care of existential restrictions $\exists r.C$ for the role names r that do not entail any role name in Σ. If $\sigma \in \mathcal{T}$, $\exists r.C \in t \in T_i$ with $\mathsf{tail}_i(\sigma) = T_i$ and $\mathcal{O} \not\models r \sqsubseteq s$ for any $s \in \Sigma$, we add $((t, \sigma), (t', p))$ to $r^{\mathcal{J}_i}$ (and all $s^{\mathcal{J}_i}$ with $\mathcal{O} \models r \sqsubseteq s$) for some $p = (T_1', T_2') \in \mathcal{B}$ and $t' \in T_i'$ with $C \in t'$ such that there are e, e' realizing t, t' in $\mathcal{I}_i$ and $(e, e') \in r^{\mathcal{I}_i}$.

The following example illustrates the construction of $\mathcal{J}_1, \mathcal{J}_2$ using the interpretations $\mathcal{I}_1, \mathcal{I}_2$ introduced in Example 5.

Example 9. *Let* $t_0 = \mathsf{tp}_\Xi(\mathcal{I}_1, a^{\mathcal{I}_1})$, $t_1 = \mathsf{tp}_\Xi(\mathcal{I}_2, b^{\mathcal{I}_2})$, *and* $t_2 = \mathsf{tp}_\Xi(\mathcal{I}_2, d)$. *We ignore the types realized by* $b^{\mathcal{I}_1}$ *in* $\mathcal{I}_1$ *and by* $a^{\mathcal{I}_2}$ *in* $\mathcal{I}_2$ *as they are not relevant for understanding the construction. Then only the mosaic* $p = (T_1, T_2)$ *with* $T_1 = \{t_0\}$ *and* $T_2 = \{t_1, t_2\}$ *remains and* $\mathcal{J}_1$ *and* $\mathcal{J}_2$ *are depicted in Figure 3.*

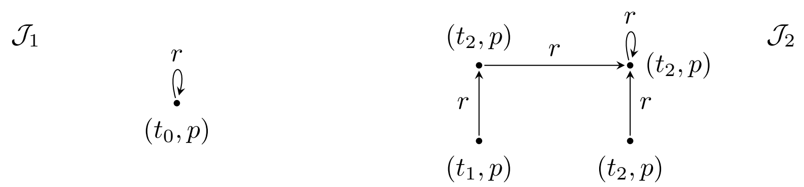

Figure 3: Interpretations $\mathcal{J}_1$ and $\mathcal{J}_2$ illustrating Example 9.

We show that $\mathcal{J}_1, \mathcal{J}_2$ are as required. First, for $i \in \{1, 2\}$, $\mathcal{J}_i \models \mathcal{O}$ follows from the definition of $\mathcal{J}_i$ and the fact that $\mathcal{I}_i \models \mathcal{O}$. Indeed, given $r \sqsubseteq s \in \mathcal{O}$, let $((t, \sigma), (t', \sigma')) \in r^{\mathcal{J}_i}$. This means that $(t, t') \in R^{r,i}_{\mathsf{tail}(\sigma),\mathsf{tail}(\sigma')}$, that is, there exist e, e' realizing t and t', respectively, with $(T_1(e), T_2(e)) = \mathsf{tail}(\sigma)$ and $(T_1(e'), T_2(e')) = \mathsf{tail}(\sigma')$, such that $(e, e') \in r^{\mathcal{I}_i}$. Since $\mathcal{I}_i \models \mathcal{O}$, we obtain that $(e, e') \in s^{\mathcal{I}_i}$ as well, and thus $(t, t') \in R^{s,i}_{\mathsf{tail}(\sigma),\mathsf{tail}(\sigma')}$, meaning that $((t, \sigma), (t', \sigma')) \in s^{\mathcal{J}_i}$. Hence, $\mathcal{J}_i \models r \sqsubseteq s$.

We next prove that, for every $(t, \sigma) \in \Delta^{\mathcal{J}_i}$ and every concept $C \in \Xi$ of depth $\leq k - |\sigma|$,

$$(t, \sigma) \in C^{\mathcal{J}_i} \text{ iff } C \in t.$$

The proof is by induction on the structure of C. We consider the case $C = \exists r.D$, where D has depth $< k - |\sigma|$. We can assume that $|\sigma| < k$, since for $|\sigma| = k$ the claim holds trivially.

($\Rightarrow$) Let $(t, \sigma) \in \exists r.D^{\mathcal{J}_i}$. Then $\exists r.D \in t$ follows by construction of $r^{\mathcal{J}_i}$ as we only have $((t, \sigma), (t', \sigma')) \in r^{\mathcal{J}_i}$ if there are e, e' realizing t, t' in $\mathcal{I}_i$ such that $(e, e') \in r^{\mathcal{I}_i}$.

($\Leftarrow$) Let $\mathsf{tail}(\sigma) = p = (T_1, T_2)$ and suppose that $\exists r.D \in t \in T_i$. We distinguish two cases.

- There exists $s \in \Sigma$ such that $\mathcal{O} \models r \sqsubseteq s$. Then there exists $q = (T_1', T_2') \in \mathcal{S}(p)$ and $t' \in T_i'$ such that $(t, t') \in R^{r,i}_{p,q}$ and $D \in t'$. We distinguish two cases.
  - t' does not contain nominals. Then we have that $((t, \sigma), (t', \sigma q)) \in r^{\mathcal{J}_i}$. By inductive hypothesis, $(t', \sigma q) \in D^{\mathcal{J}_i}$, and thus $(t, \sigma) \in \exists r.D^{\mathcal{J}_i}$.
  - t' contains a nominal. Then we have that $((t, \sigma), (t', q)) \in r^{\mathcal{J}_i}$. By inductive hypothesis, $(t', q) \in D^{\mathcal{J}_i}$, hence $(t, \sigma) \in \exists r.D^{\mathcal{J}_i}$.
- For every $s \in \Sigma$, $\mathcal{O} \not\models r \sqsubseteq s$. By definition of $\mathcal{J}_i$, we have $((t, \sigma), (t', q)) \in r^{\mathcal{J}_i}$, for some $q = (T_1', T_2') \in \mathcal{B}$ and $t' \in T_i'$ such that $D \in t'$. By inductive hypothesis, $(t', q) \in D^{\mathcal{J}_i}$. Thus, $(t, \sigma) \in \exists r.D^{\mathcal{J}_i}$.

Next observe that the relation

$$S = \{((t, \sigma), (t', \sigma')) \in \Delta^{\mathcal{J}_1} \times \Delta^{\mathcal{J}_2} \mid \mathsf{tail}(\sigma) = \mathsf{tail}(\sigma')\}$$

is an $\mathcal{ALCHO}(\Sigma)$-bisimulation. Indeed, for $((t, \sigma), (t', \sigma')) \in S$, we have the following.

[AtomC] Let $(t, \sigma) \in A^{\mathcal{J}_1}$ and $A \in \Sigma$. By definition of $\mathcal{J}_1$, we have that $(t, \sigma) \in A^{\mathcal{J}_1}$ iff $A \in t \in \mathsf{tail}_1(\sigma)$, and thus $A \in t' \in \mathsf{tail}_2(\sigma) = \mathsf{tail}_2(\sigma')$, by definition of mosaics. But then $(t', \sigma') \in A^{\mathcal{J}_2}$. The converse direction is analogous.

[AtomI] Let $(t, \sigma) = a^{\mathcal{J}_1}$ and $a \in \Sigma$. By definition of $\mathcal{J}_1$, $(t, \sigma) = a^{\mathcal{J}_1}$ iff $\{a\} \in t \in \mathsf{tail}_1(\sigma)$, and thus $\{a\} \in t' \in \mathsf{tail}_2(\sigma) = \mathsf{tail}_2(\sigma')$, by definition of mosaics. But then $(t', \sigma') = a^{\mathcal{J}_2}$.

[Forth] Suppose that $((t, \sigma), (\hat{t}, \hat{\sigma})) \in r^{\mathcal{J}_1}$ with $r \in \Sigma$.

First, consider the case with $|\sigma|, |\sigma'| < k$. We have two possibilities.

- $\hat{t}$ does not contain nominals. The following proof is illustrated in Figure 5. There is a mosaic p with $\hat{\sigma} = \sigma p$, and from $((t, \sigma), (\hat{t}, \sigma p)) \in r^{\mathcal{J}_1}$ we obtain $(t, \hat{t}) \in R^{r,1}_{\mathsf{tail}(\sigma),p}$. This means that there exist $d, \hat{d}$ realizing t and $\hat{t}$, respectively, with $(T_1(d), T_2(d)) = \mathsf{tail}(\sigma)$ and $(T_1(\hat{d}), T_2(\hat{d})) = p$, such that $(d, \hat{d}) \in r^{\mathcal{I}_1}$. As $t' \in T_2(d)$, there exists $e \in \Delta^{\mathcal{I}_2}$ with $\mathcal{I}_1, d \sim_{\mathcal{ALCHO},\Sigma} \mathcal{I}_2, e$ and e realizes t'. By the definition of bisimulations, there exists $\hat{e}$ with $(e, \hat{e}) \in r^{\mathcal{I}_2}$ and $\mathcal{I}_1, \hat{d} \sim_{\mathcal{ALCHO},\Sigma} \mathcal{I}_2, \hat{e}$. Assume that $\hat{e}$ realizes $\hat{t}'$. Then $\hat{t}' \in T_2(\hat{d})$ and $(\hat{t}, \hat{t}') \in R^{r,2}_{\mathsf{tail}(\sigma'),p}$. Now we consider again two possibilities.
  - $\hat{t}'$ does not contain nominals. Then from $(t', \hat{t}') \in R^{r,2}_{\mathsf{tail}(\sigma'),p}$ we obtain $((t', \sigma'), (\hat{t}', \sigma' p)) \in r^{\mathcal{J}_2}$. Since $\mathsf{tail}(\sigma p) = \mathsf{tail}(\sigma' p)$, we also obtain $((\hat{t}, \sigma p), (\hat{t}', \sigma' p)) \in S$.

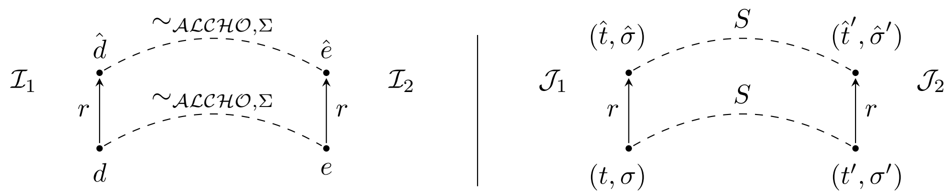

Figure 4: Proof step to show that S satisfies [Forth], with $\hat{\sigma} = \sigma p$ and $\hat{\sigma}' = \sigma' p$ (if $\hat{t}'$ does not contain nominals) and $\hat{\sigma}' = p$ (if $\hat{t}'$ contains nominals).

   – $\hat{t}'$ contains nominals. Then from $(t', \hat{t}') \in R^{r,2}_{\mathsf{tail}(\sigma'),p}$ we obtain $((t', \sigma'), (\hat{t}', p)) \in r^{\mathcal{J}_2}$. Since $\mathsf{tail}(\sigma p) = p$, we get $((\hat{t}, \sigma p), (\hat{t}', p)) \in S$ as well.

  In both cases, we obtain some $(\hat{t}', \hat{\sigma}')$ with $((t', \sigma'), (\hat{t}', \hat{\sigma}')) \in r^{\mathcal{J}_2}$ and $((\hat{t}, \hat{\sigma}), (\hat{t}', \hat{\sigma}')) \in S$, as required.

- $\hat{t}$ contains nominals. In this case, $\hat{\sigma} = p$ for some mosaic p, and from $((t, \sigma), (\hat{t}, p)) \in r^{\mathcal{J}_1}$ we obtain $(t, \hat{t}) \in R^{r,1}_{\mathsf{tail}(\sigma),p}$. Now we can reason as above.

Now consider the case with $|\sigma| = k$ and $|\sigma'| < k$. As $\mathsf{tail}(\sigma) = \mathsf{tail}(\sigma')$, there exists σ'' such that $|\sigma''| < k$ and $\mathsf{tail}(\sigma'') = \mathsf{tail}(\sigma)$ and the r-successors of any node of the form (t, σ) are exactly the r-successors of (t, σ''), and thus to show [Forth] one can proceed as above. The same argument applies if $|\sigma| < k$ and $|\sigma'| = k$ and if $|\sigma| = |\sigma'| = k$ and there exists σ'' with $\mathsf{tail}(\sigma'') = \mathsf{tail}(\sigma') = \mathsf{tail}(\sigma)$ and $|\sigma''| < k$. Finally, if $|\sigma| = |\sigma'| = k$ but there does not exist any σ'' with $\mathsf{tail}(\sigma'') = \mathsf{tail}(\sigma') = \mathsf{tail}(\sigma)$ and $|\sigma''| < k$, then there are no r-successors to consider.

[Back] Dual to [Forth].

Observe that the models $\mathcal{J}_i$, $i = 1, 2$, are at most exponential in the size of $\mathcal{O}, C_1, C_2$. Moreover, we have $(T_1(d_1), (T_2(d_1)) \in \mathcal{B}$ and so $(\mathsf{tp}_\Xi(\mathcal{I}_1, d_1), T_1(d_1)) \in C_1^{\mathcal{J}_1}$, $(\mathsf{tp}_\Xi(\mathcal{I}_2, d_2), T_2(d_1)) \in (\neg C_2)^{\mathcal{J}_2}$, and

$$((\mathsf{tp}_\Xi(\mathcal{I}_1, d_1), T_1(d_1)), (\mathsf{tp}_\Xi(\mathcal{I}_2, d_2), T_2(d_1)) \in S,$$

as required.

We next consider the case with inverse roles, but without nominals. In this case we let $\mathcal{B}$ be some minimal set of mosaics defined by $\mathcal{I}_1, \mathcal{I}_2$ containing $(T_1(d_1), T_2(d_1))$ and such that for every type t realized in $\mathcal{I}_i$ there exists $(T_1, T_2) \in \mathcal{B}$ with $t \in T_i$. We extend the relations $R^{s,i}_{p,q}$ defined previously to inverse roles s in the obious way and select for any mosaic $p = (T_1, T_2)$ and any $\exists s.C \in t \in T_i$ such that there exists a Σ-role r with $\mathcal{O} \models s \sqsubseteq r$ a mosaic $q = (T_1', T_2')$ such that $(t, t') \in R^{s,i}_{p,q}$ and $C \in t'$ and denote the resulting set by $\mathcal{S}(p)$.

Form again the set $\mathcal{T}$ of sequences

$$\sigma = p_0 \cdots p_j = (T_1^0, T_2^0) \cdots (T_1^j, T_2^j)$$

with $j \leq k$, $p_0 \in \mathcal{B}$ and $p_{i+1} \in \mathcal{S}(p_i)$ for $i < j$. Let $\mathsf{tail}(\sigma) = p_j$ and $\mathsf{tail}_i(\sigma) = T_i^j$. We next define the domain of $\mathcal{J}_1$ and $\mathcal{J}_2$ as

$$\Delta^{\mathcal{J}_i} = \{(t, \sigma) \mid \sigma \in \mathcal{T}, t \in \mathsf{tail}_i(\sigma)\}$$

We define interpretations $\mathcal{J}_1, \mathcal{J}_2$ in the expected way.

- For any concept name A, $(t, \sigma) \in A^{\mathcal{J}_i}$ iff $A \in t$;
- Let r be a role name. Then we let for $\sigma p \in \mathcal{T}$,
  - $((t, \sigma), (t', \sigma p)) \in r^{\mathcal{J}_i}$ if $(t, t') \in R^{r,i}_{\text{tail}(\sigma),p}$;
  - $((t', \sigma p), (t, \sigma)) \in r^{\mathcal{J}_i}$ if $(t, t') \in R^{r^-,i}_{\text{tail}(\sigma),p}$.
- We still have to take care of existential restrictions $\exists r.C$ with r a role that does not entail any Σ-role. If $\sigma \in \mathcal{T}$, $\exists r.C \in t \in T_i$ with $\text{tail}_i(\sigma) = T_i$ and $\mathcal{O} \not\models r \sqsubseteq s$ for any Σ-role s, we add $((t, \sigma), (t', p))$ to $r^{\mathcal{J}_i}$ (and all $s^{\mathcal{J}_i}$ with $\mathcal{O} \models r \sqsubseteq s$) for some $p = (T_1', T_2') \in \mathcal{B}$ and $t' \in T_i'$ with $C \in t'$ such that there are e, e' realizing t, t' in $\mathcal{I}_i$ and $(e, e') \in r^{\mathcal{I}_i}$.

The fact that $\mathcal{J}_i \models \mathcal{O}$, for $i \in \{1, 2\}$, is proved similarly to the case with nominals. One can also prove again by induction on the structure of C that for every $(t, \sigma) \in \Delta^{\mathcal{J}_i}$ and every $C \in \Xi$ of depth $\leq k - |\sigma|$,

$$(t, \sigma) \in C^{\mathcal{J}_i} \text{ iff } C \in t.$$

Next we observe that the relation

$$S = \{((t, \sigma), (t', \sigma)) \in \Delta^{\mathcal{J}_1} \times \Delta^{\mathcal{J}_2} \mid \sigma \in \mathcal{T}\}$$

is an $\mathcal{ALCHI}(\Sigma)$-bisimulation. Indeed, it can be seen, similarly to the case with nominals, that S satisfies [AtomC]. We now give a proof of [Forth]. We provide the proof for role names; the proof for inverse roles is similar.

[Forth] Let $((t, \sigma), (t', \sigma)) \in S$ and $((t, \sigma), (\hat{t}, \hat{\sigma})) \in r^{\mathcal{J}_1}$. We distinguish two cases. Assume first that there exists a mosaic p with $\hat{\sigma} = \sigma p$. Then $(t, \hat{t}) \in R^{r,1}_{\text{tail}(\sigma),p}$. Thus, there exist $d, \hat{d}$ realizing $t, \hat{t}$, respectively, such that $(T_1(d), T_2(d)) = \text{tail}(\sigma)$, $(T_1(\hat{d}), T_2(\hat{d})) = p$, and $(d, \hat{d}) \in r^{\mathcal{I}_1}$. Since $((t, \sigma), (t', \sigma)) \in S$, there exists e realizing t' such that $\mathcal{I}_1, d \sim_{\mathcal{ALCHI},\Sigma} \mathcal{I}_2, e$. By bisimilarity of d and e, we also have some $\hat{e} \in \Delta^{\mathcal{I}_2}$ such that $(e, \hat{e}) \in r^{\mathcal{I}_2}$ and $\mathcal{I}_1, \hat{d} \sim_{\mathcal{ALCHI},\Sigma} \mathcal{I}_2, \hat{e}$, with $\hat{e}$ realizing some $\hat{t}'$. Hence, $(t', \hat{t}') \in R^{r,2}_{\text{tail}(\sigma),p}$, and it follows that $((t', \sigma), (\hat{t}', \sigma p)) \in r^{\mathcal{J}_2}$. Moreover, $((\hat{t}, \sigma p), (\hat{t}', \sigma p)) \in S$.

Assume now that $\sigma = \hat{\sigma} p$ for some mosaic p. Then $(\hat{t}, t) \in R^{r^-,1}_{\text{tail}(\hat{\sigma}),p}$. Thus, there exist $\hat{d}, d$ realizing $\hat{t}, t$, respectively, such that $(T_1(\hat{d}), T_2(\hat{d})) = \text{tail}(\hat{\sigma})$, $(T_1(d), T_2(d)) = p$, and $(\hat{d}, d) \in (r^-)^{\mathcal{I}_1}$. Since $((t, \sigma), (t', \sigma)) \in S$, there exists e realizing t' such that $\mathcal{I}_1, d \sim_{\mathcal{ALCHI},\Sigma} \mathcal{I}_2, e$. By bisimilarity of d and e, we also have some $\hat{e} \in \Delta^{\mathcal{I}_2}$ such that $(\hat{e}, e) \in (r^-)^{\mathcal{I}_2}$ and $\mathcal{I}_1, \hat{d} \sim_{\mathcal{ALCHI},\Sigma} \mathcal{I}_2, \hat{e}$, with $\hat{e}$ realizing some $\hat{t}'$. Hence, $(\hat{t}', t') \in R^{r^-,2}_{\text{tail}(\hat{\sigma}),p}$, and it follows that $((t', \sigma), (\hat{t}', \hat{\sigma})) \in r^{\mathcal{J}_2}$. Moreover, $((\hat{t}, \hat{\sigma}), (\hat{t}', \hat{\sigma})) \in S$.

Observe that again the models $\mathcal{J}_i$, $i = 1, 2$, are of at most exponential size in the size of $\mathcal{O}, C_1, C_2$. We also have $(T_1(d_1), (T_2(d_1)) \in \mathcal{B}$ and so $(\text{tp}_\Xi(\mathcal{I}_1, d_1), T_1(d_1)) \in C_1^{\mathcal{J}_1}$, $(\text{tp}_\Xi(\mathcal{I}_2, d_2), T_2(d_1)) \in (\neg C_2)^{\mathcal{J}_2}$, and

$$((\text{tp}_\Xi(\mathcal{I}_1, d_1), T_1(d_1)), (\text{tp}_\Xi(\mathcal{I}_2, d_2), T_2(d_1)) \in S,$$

as required. ❏

Now for the lower bound. We show that it is NExpTime-hard to decide joint consistency of $\mathcal{L}$-concepts $C \sqcap C_0$ and $C \sqcap \neg C_0$ (under an ontology containing RIs) modulo $\mathcal{L}(\Sigma)$-bisimulations and then employ Theorem 8. We encode the following problem. An *(exponential torus) tiling problem* P is a triple (T, H, V), where $T = \{0, \ldots, k\}$ is a finite set of *tile types* and $H, V \subseteq T \times T$ represent the *horizontal and vertical matching conditions*. An *initial condition* for P takes the form $c = (c_0, \ldots, c_{n-1}) \in T^n$. A mapping $\tau : \{0, \ldots, 2^n - 1\} \times \{0, \ldots, 2^n - 1\} \to T$ is a *solution* for P given c if for all $x, y < 2^n$, the following holds (where $\oplus_i$ denotes addition modulo i):

- if $\tau(x, y) = t_1$ and $\tau(x \oplus_{2^n} 1, y) = t_2$, then $(t_1, t_2) \in H$
- if $\tau(x, y) = t_1$ and $\tau(x, y \oplus_{2^n} 1) = t_2$, then $(t_1, t_2) \in V$
- $\tau(i, 0) = c_i$ for all $i < n$.

It is well-known that there exists a tiling problem $P = (T, H, V)$ such that, given an initial condition c, it is NExpTime-complete to decide whether there exists a solution for P given c. For the following constructions, we fix such a P.

For our reduction for $\mathcal{ALCO}$ we set

$$C_0 = \exists r^{2n}.\{a\} \sqcap \forall r^{2n}.\{a\}$$

with $a \notin \Sigma$ and $r \in \Sigma$. In addition to r, Σ contains concept names $B_0, \ldots, B_{2n-1}$ and $\overline{B}_0, \ldots, \overline{B}_{2n-1}$ that serve as bits in the binary representation of *grid positions* (i, j) with $0 \leq i, j \leq 2^n - 1$, where bits 0 to $n-1$ represent the horizontal position i and bits n to $2n-1$ the vertical position j, and concept names $T_0, \ldots, T_k$ representing tile types. We also use the following concept names that are not in Σ: another four sets of concepts names $A_0, \ldots, A_{2n-1}$, $\overline{A}_0, \ldots, \overline{A}_{2n-1}$ and $V_0, \ldots, V_{2n-1}$, $\overline{V}_0, \ldots, \overline{V}_{2n-1}$ with $V \in \{X, Y, Z\}$ that also serve as bits in the binary representation of grid position (i, j) with $0 \leq i, j \leq 2^n - 1$, and concept names $R_0, \ldots, R_{2n}$, M, M_1, and M_2. We now define the concept C as a conjunction of several concepts. The first conjunct is

$$\neg C_0 \sqcap \exists r^{2n}.\top \to R_0.$$

Intuitively, R_0 generates a binary r-tree of depth $2n$ with R_i true at level i for $0 \leq i \leq 2^{2n}$ and each leaf represents a grid position (i, j) using the concept names $A_i/\overline{A}_i$. To achieve this let C contain the following conjuncts for generating the binary tree:

$$\forall r^{<2n}. \bigsqcap_{0 \leq i < 2n} \big(R_i \to (\exists r.(A_i \sqcap R_{i+1}) \sqcap \exists r.(\overline{A}_i \sqcap R_{i+1}))\big)$$

$$\forall r^{<2n}. \bigsqcap_{1 \leq i < 2n} (A_i \to \forall r.A_i) \sqcap (\overline{A}_i \to \forall r.\overline{A}_i)$$

(Here and in what follows we define inductively $\forall r^{<1}.D := D$ and $\forall r^{<m+1}.D := \forall r^m.D \sqcap \forall r^{<m}.D$.)

We next express using additional conjuncts of C that any leaf d representing (i, j) using $A_i/\overline{A}_i$ has the following properties (A) – (D):

(A) d has an r-successor representing (i, j) using $B_i/\overline{B}_i$ with a tile type $T_{(i,j)}$ true in it; moreover, no r-successor of d representing (i, j) satisfies a tile type different from $T_{(i,j)}$. This is achieved using the marker M which holds in exactly those r-successors of d that represent (i, j) using $B_i/\overline{B}_i$. The latter condition is expressed using the counter $X_i/\overline{X}_i$ which represents (i, j) on all r-successors of d. In detail, we add the following conjuncts to C:

$$\forall r^{2n}.\exists r.M$$

$$\forall r^{2n}.\big(\bigsqcap_{i<2n}(A_i \rightarrow \forall r.X_i) \sqcap (\overline{A}_i \rightarrow \forall r.\overline{X}_i)\big)$$

$$\forall r^{2n+1}.\big(M \leftrightarrow \bigsqcap_{i<2n}(X_i \leftrightarrow B_i) \sqcap (\overline{X}_i \leftrightarrow \overline{B}_i)\big)$$

$$\forall r^{2n}.\big(\forall r.(M \sqcap \bigsqcup_{i\leq k} T_i) \sqcap \bigsqcap_{i\leq k} \exists r.(M \sqcap T_i) \rightarrow \forall r.(M \rightarrow T_i)\big)$$

$$\forall r^{2n+1}.\bigsqcap_{i\neq j} \neg(T_i \sqcap T_j)$$

(B) d has an r-successor representing $(i \oplus_{2^n} 1, j)$ using $B_i/\overline{B}_i$ with a tile type $T^{\text{right}}_{(i,j)}$ true in it such that $(T_{(i,j)}, T^{\text{right}}_{(i,j)}) \in H$; moreover, no r-successor of d representing $(i \oplus_{2^n} 1, j)$ satisfies a tile type different from $T^{\text{right}}_{(i,j)}$. This is achieved similarly to (A) using the marker M_1 which holds in exactly those r-successors of d that represent $(i \oplus_{2^n} 1, j)$ using $B_i/\overline{B}_i$. The latter condition is expressed using the counter $Y_i/\overline{Y}_i$ which represents $(i \oplus_{2^n} 1, j)$ on all r-successors of d. The implementation of these conditions is similar to (A) and omitted.

(C) d has an r-successor representing $(i, j \oplus_{2^n} 1)$ using $B_i/\ \overline{B}_i$ with a tile type $T^{\text{up}}_{(i,j)}$ true in it such that $(T_{(i,j)}, T^{\text{up}}_{(i,j)}) \in V$; moreover, no r-successor of d representing $(i, j \oplus_{2^n} 1)$ satisfies a tile type different from $T^{\text{up}}_{(i,j)}$. This is achieved similarly to (A) using the marker M_2 which holds in exactly those r-successors of d that represent $(i, j \oplus_{2^n} 1)$ using $B_i/\overline{B}_i$. The latter condition is expressed using the counter $Z_i/\overline{Z}_i$ which represents $(i, j \oplus_{2^n} 1)$ on all r-successors of d. The implementation is again similar to (A) and omitted.

(D) The initial condition holds, that is $T_{(i,0)} = c_i$ for $i < n$. To this end we add the conjuncts

$$\forall r^{2n}.(A = (i, 0) \rightarrow (\forall r.(M \rightarrow c_i)))$$

for $i < n$, where $A = (i, 0)$ stands for the representation of $(i, 0)$ using $A_i/\overline{A}_i$; for instance, $A = (0, 0)$ stands for $\bigsqcap_{0\leq i<2n} \overline{A}_i$.

Claim. There exist $\mathcal{I}_1, d_1 \sim_{\mathcal{ALCO},\Sigma} \mathcal{I}_2, d_2$ with $d_1 \in (C \sqcap C_0)^{\mathcal{I}_1}$ and $d_2 \in (C \sqcap \neg C_0)^{\mathcal{I}_2}$ iff P has a solution.

Proof of Claim. Observe that if $\mathcal{I}_1, d_1 \sim_{\mathcal{ALCO},\Sigma} \mathcal{I}_2, d_2$ with $d_1 \in (C \sqcap C_0)^{\mathcal{I}_1}$ and $d_2 \in (C \sqcap \neg C_0)^{\mathcal{I}_2}$, then there are nodes $e_{(i,j)}$, $0 \leq i, j \leq 2^n - 1$ such that

$$\mathcal{I}_1, a^{\mathcal{I}_1} \sim_{\mathcal{ALCO},\Sigma} \mathcal{I}_2, e_{(i,j)}$$

and $e_{(i,j)}$ has (at least) three r-successors satisfying Conditions (A) to (D). By Σ-bisimilarity and since $r \in \Sigma$, all $e_{(i,j)}$ have r-successors satisfying the same concept names in Σ. Hence, since the concept names $B_i/\overline{B}_i$ and T_i are in Σ, for every grid position (i,j) every $e_{(i',j')}$ has an r-successor representing (i,j) using $B_i/\overline{B}_i$ and all r-successors representing (i,j) using $B_i, \overline{B}_i$ satisfy the same tile type $T_{(i,j)}$. Moreover, $T_{(i \oplus_{2^n} 1, j)} = T^{\text{right}}_{(i,j)}$ and $T_{(i, j \oplus_{2^n} 1)} = T^{\text{up}}_{(i,j)}$. It follows that the mapping τ defined by setting $\tau(i,j) = T_{(i,j)}$ is a solution.

Conversely, assume that P has a solution τ. The definition of interpretations $\mathcal{I}_1$ and $\mathcal{I}_2$ with nodes d_1 and d_2 such that $\mathcal{I}_1, d_1 \sim_{\mathcal{ALCO},\Sigma} \mathcal{I}_2, d_2$ with $d_1 \in (C \sqcap C_0)^{\mathcal{I}_1}$ and $d_2 \in (C \sqcap \neg C_0)^{\mathcal{I}_2}$ is rather straightforward. An abtract version is depicted in Figure 5. Note that we omit the counters and that $\{a\}$ and *all* leaves at level R_{2n} have an r-successor representing (i,j) using $B_i/\overline{B}_i$ for $0 \leq i,j \leq 2^n - 1$.

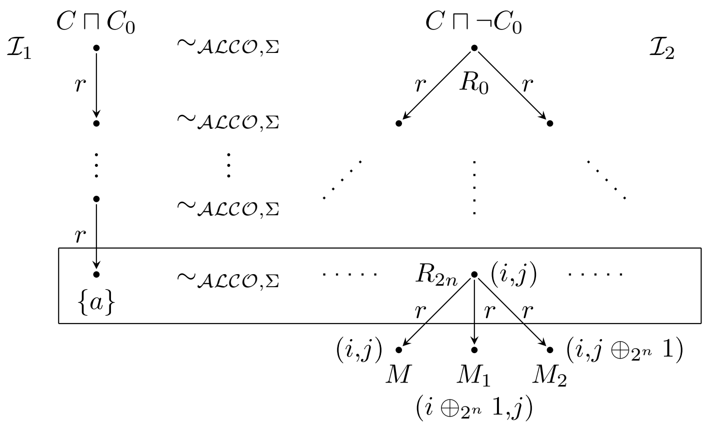

Figure 5: Models $\mathcal{I}_1$ and $\mathcal{I}_2$ of $C \sqcap C_0$ and $C \sqcap \neg C_0$, given a solution for the tiling problem.

We come to the lower bound for $\mathcal{ALCH}$ and $\mathcal{ALCHI}$. Let

$$\mathcal{O} = \{r \sqsubseteq r_1, r \sqsubseteq r_2, r_1 \sqsubseteq v, r_2 \sqsubseteq v\},$$

and Σ contains r_1, r_2 but not r nor v. We set $C_0 = \exists r^{2n}.\top$ and start the definition of the concept C with the conjunct

$$\neg \exists r^{2n}.\top \sqcap \exists v^{2n}.\top \rightarrow R_0$$

where R_0 generates a binary r_1/r_2-tree whose leaves represent the grid positions (i,j) with $0 \leq i,j \leq 2^n - 1$. The construction of C is similar to the proof for $\mathcal{ALCO}$, the only difference is that the role of the individual name a is now played by a node reachable along r in $2n$-steps and that we use two role names, r_1 and r_2, to build the binary tree. The role name v is introduced to ensure that we can reach all leaves of the r_1/r_2-tree using v^{2n}. In more detail, in addition to r_1 and r_2, Σ contains exactly the same concept names as in the $\mathcal{ALCO}$ proof and we also use the same concept names not in Σ. To generate the binary tree we now use the following conjuncts of C:

$$\forall v^{<2n}. \bigsqcap_{0 \leq i < 2n} \big(R_i \rightarrow (\exists r_1.(A_i \sqcap R_{i+1}) \sqcap \exists r_2.(\overline{A}_i \sqcap R_{i+1}))\big)$$

$$\forall v^{<2n}. \bigsqcap_{0 \leq i < 2n} \big(R_i \rightarrow (\forall r_1.(A_i \sqcap \neg \overline{A}_i \sqcap R_{i+1}) \sqcap \forall r_2.(\overline{A}_i \sqcap \neg A_i \sqcap R_{i+1}))\big)$$

$$\forall v^{<2n}. \bigsqcap_{0 < i < 2n} (A_i \rightarrow \forall r_1.A_i \sqcap \forall r_2.A_i) \sqcap (\overline{A}_i \rightarrow \forall r_1.\overline{A}_i \sqcap \forall r_2.\overline{A}_i)$$

We continue the definition of C in exactly the same way as for $\mathcal{ALCO}$ except that we use $\forall v^{2n}$ to reach the leaves of the tree and r_1-successors of the leaves to encode a solution of the tiling problem. One can then easily prove the following.

Claim. There exist $\mathcal{I}_1, d_1 \sim_{\mathcal{ALCH},\Sigma} \mathcal{I}_2, d_2$ with $\mathcal{I}_1, \mathcal{I}_2$ models of $\mathcal{O}$, $d_1 \in (C \sqcap \exists r^{2n}.\top)^{\mathcal{I}_1}$, and $d_2 \in (C \sqcap \neg \exists r^{2n}.\top)^{\mathcal{I}_2}$ iff P has a solution.

Proof of Claim. Observe that if $\mathcal{I}_1, d_1 \sim_{\mathcal{ALCH},\Sigma} \mathcal{I}_2, d_2$ with $\mathcal{I}_1, \mathcal{I}_2$ models of $\mathcal{O}$, $d_1 \in (C \sqcap \exists r^{2n})^{\mathcal{I}_1}$, and $d_2 \in (C \sqcap \neg \exists r^{2n})^{\mathcal{I}_2}$, then there exists a node e reachable from d_1 along an r-chain of length $2n$ in $\mathcal{I}_1$ and nodes $e_{(i,j)}$, $0 \leq i, j \leq 2^n - 1$, reachable from d_2 along a v-chain of length $2n$ in $\mathcal{I}_2$ such that $\mathcal{I}_1, e \sim_{\mathcal{ALCH},\Sigma} \mathcal{I}_2, e_{(i,j)}$. Now the proof is essentially the same as for $\mathcal{ALCO}$.

To prove the claim above for $\mathcal{ALCHI}$ the construction of the interpretations $\mathcal{I}_1$ and $\mathcal{I}_2$ has to ensure $\mathcal{ALCHI}(\Sigma)$-bisimilarity. This is not the case if one just takes tree-shaped interpretations as some nodes only have r_1-predecessors and others only r_2-predecessors. One therefore has to attach new predecessors to the tree-shaped interpretations, together with copies of the original tree-shaped interpretations; this process continues in the obvious way, resulting in $\mathcal{ALCHI}(\Sigma)$-bisimilar pointed interpretations.

6. The Computation Problem

We have presented algorithms for *deciding* the existence of interpolants, but these algorithms (and their correctness proofs) do not give immediately rise to a way of *computing* interpolants in case they exist. Intuitively, this is due to the fact that compactness is used in the proof of the model-theoretic characterization in Theorem 7. The main result in this section is the following.

Theorem 11. *Let $\mathcal{L}$ be a DL in $\mathsf{DL}_{\mathsf{nr}}$ that does not contain nominals, and let $\mathcal{O}$ be an $\mathcal{L}$-ontology, C_1, C_2 be $\mathcal{L}$-concepts, and Σ be a signature. Then, if there is an $\mathcal{L}(\Sigma)$-interpolant for $C_1 \sqsubseteq C_2$ under $\mathcal{O}$, we can compute the DAG representation of an $\mathcal{L}(\Sigma)$-interpolant in time $2^{2^{p(n)}}$ where p is a polynomial and $n = ||\mathcal{O}|| + ||C_1|| + ||C_2||$.*

Note that this implies that the DAG representation is also of double exponential size, and that a formula representation of the interpolant can be computed in triple exponential time. Moreover, this also allows us to compute explicit definitions since, given $\mathcal{O}$, C, and Σ, any $\mathcal{L}(\Sigma)$-interpolant for $C_\Sigma \sqsubseteq C$ under $\mathcal{O} \cup \mathcal{O}_\Sigma$ is an explicit $\mathcal{L}(\Sigma)$-definition of C under $\mathcal{O}$, where $\mathcal{O}_\Sigma$ and C_Σ are obtained from $\mathcal{O}$ and C by replacing all symbols not in Σ by fresh symbols.

Let $\mathcal{L}$, $\mathcal{O}$, C_1, C_2, and Σ be as in Theorem 11. The computation of the $\mathcal{L}(\Sigma)$-interpolant (if it exists) is based on a *mosaic elimination procedure* for deciding joint consistency, which is a simplified variant of a procedure that was presented in [13].[1] As in Section 5, a *mosaic* is a

[1] The procedure in [13] decides a slightly more general variant of joint consistency.

pair (T_1, T_2) with T_1, T_2 sets of Ξ-types, where $\Xi = \text{sub}(\mathcal{O}, C_1, C_2)$. We denote with $\text{Tp}(\Xi)$ the set of all Ξ-types. The aim of the mosaic elimination procedure is to determine all pairs $(T_1, T_2) \in 2^{\text{Tp}(\Xi)} \times 2^{\text{Tp}(\Xi)}$ such that all $t \in T_1 \cup T_2$ can be realized in mutually $\mathcal{L}(\Sigma)$-bisimilar elements of models of $\mathcal{O}$. In order to formulate the elimination conditions, we need some preliminary notions. Throughout the rest of the section, we treat the universal role u as a role name contained in Σ, in case $\mathcal{L}$ allows the universal role. Note that u^- is equivalent to u, and that $\mathcal{O} \models r \sqsubseteq u$, for every role r.

Let t_1, t_2 be Ξ-types. We call t_1, t_2 *u-equivalent* if for every $\exists u.C \in \Xi$, we have $\exists u.C \in t_1$ iff $\exists u.C \in t_2$. This condition is trivial if $\mathcal{L}$ does not use allow the universal role. For a role r, we call t_1, t_2 *r-coherent for $\mathcal{O}$*, in symbols $t_1 \rightsquigarrow_{r,\mathcal{O}} t_2$, if t_1, t_2 are u-equivalent and the following conditions hold for all roles s with $\mathcal{O} \models r \sqsubseteq s$: (1) if $\neg\exists s.C \in t_1$, then $C \notin t_2$ and (2) if $\neg\exists s^-.C \in t_2$, then $C \notin t_1$. Note that $t \rightsquigarrow_{r,\mathcal{O}} t'$ iff $t' \rightsquigarrow_{r^-,\mathcal{O}} t$. We lift the definition of r-coherence from types to mosaics $(T_1, T_2), (T_1', T_2')$. We call (T_1, T_2), (T_1', T_2') *r-coherent*, in symbols $(T_1, T_2) \rightsquigarrow_r (T_1', T_2')$, if for $i = 1, 2$:

- for every $t \in T_i$ there exists a $t' \in T_i'$ such that $t \rightsquigarrow_{r,\mathcal{O}} t'$, and
- if $\mathcal{L}$ allows for inverse roles, then for every $t' \in T_i'$, there is a $t \in T_i$ such that $t \rightsquigarrow_{r,\mathcal{O}} t'$.

Note that $(T_1, T_2) \rightsquigarrow_r (T_1', T_2')$ iff $(T_1', T_2') \rightsquigarrow_{r^-} (T_1, T_2)$ if $\mathcal{L}$ allows for inverse roles.

Let $\mathcal{S} \subseteq 2^{\text{Tp}(\Xi)} \times 2^{\text{Tp}(\Xi)}$. We call $(T_1, T_2) \in \mathcal{S}$ *bad* if it violates one of the following conditions.

1. *Σ-concept name coherence*: $A \in t$ iff $A \in t'$, for every concept name $A \in \Sigma$ and any $t, t' \in T_1 \cup T_2$;
2. *Existential saturation*: for $i = 1, 2$ and $\exists r.C \in t \in T_i$, there exists $(T_1', T_2') \in \mathcal{S}$ such that (1) there exists $t' \in T_i'$ with $C \in t'$ and $t \rightsquigarrow_{r,\mathcal{O}} t'$ and (2) if $\mathcal{O} \models r \sqsubseteq s$ for a Σ-role s, then $(T_1, T_2) \rightsquigarrow_s (T_1', T_2')$.

The mosaic elimination procedure is now as follows. We start with the set $\mathcal{S}_0$ of all mosaics $(T_1, T_2) \in 2^{\text{Tp}(\Xi)} \times 2^{\text{Tp}(\Xi)}$ such that, for $i = 1, 2$, T_i contains only types that are realizable in some model of $\mathcal{O}$. Then obtain, for $i \geq 0$, $\mathcal{S}_{i+1}$ from $\mathcal{S}_i$ by eliminating all mosaics (T_1, T_2) that are bad in $\mathcal{S}_i$. Let $\mathcal{S}^*$ be where the sequence stabilizes. This elimination procedure decides joint consistency (and thus interpolant existence via Theorem 7) in the following sense.

Lemma 12. *The following conditions are equivalent:*

1. *$C_1, \neg C_2$ are jointly consistent under $\mathcal{O}$ modulo $\mathcal{L}(\Sigma)$-bisimulations;*
2. *there exists $(T_1, T_2) \in \mathcal{S}^*$ and Ξ-types $t_1 \in T_1, t_2 \in T_2$ with $C_1 \in t_1$ and $\neg C_2 \in t_2$.*

Proof. "1 $\Rightarrow$ 2". Let $\mathcal{I}_1, d_1 \sim_{\mathcal{L},\Sigma} \mathcal{I}_2, d_2$ for models $\mathcal{I}_1$ and $\mathcal{I}_2$ of $\mathcal{O}$ such that d_1, d_2 realize Ξ-types t_1, t_2 and $C_1 \in t_1, C_2 \notin t_2$.

Define $\mathcal{S}$ by setting $(T_1, T_2) \in \mathcal{S}$ if there is $d \in \Delta^{\mathcal{I}_i}$ for some $i \in \{1, 2\}$ such that

$$T_j = \{\text{tp}_\Xi(\mathcal{I}_j, e) \mid e \in \Delta^{\mathcal{I}_j}, \mathcal{I}_i, d \sim_{\mathcal{L},\Sigma} \mathcal{I}_j, e\},$$

for $j = 1, 2$. It is routine to show that no (T_1, T_2) in $\mathcal{S}$ is bad, thus $\mathcal{S} \subseteq \mathcal{S}^*$. Clearly, $\mathcal{S}^*$ satisfies Condition 2.

"$2 \Rightarrow 1$". Suppose there exist $(S_1, S_2) \in \mathcal{S}^*$ and Ξ-types $s_1 \in S_1, s_2 \in S_2$ with $C_1 \in s_1$ and $\neg C_2 \in s_2$. Let $\mathcal{I}_i$, for $i = 1, 2$ be interpretations defined by setting:

$$
\begin{aligned}
\Delta^{\mathcal{I}_i} &:= \{(t, (T_1, T_2)) \mid (T_1, T_2) \in \mathcal{S}^*, (S_1, S_2) \leadsto_u (T_1, T_2), t \in T_i, \text{ and } t, s_i \text{ are } u\text{-equivalent}\} \\
r^{\mathcal{I}_i} &:= \{((t, p), (t', p')) \in \Delta^{\mathcal{I}_i} \times \Delta^{\mathcal{I}_i} \mid t \leadsto_{r,\mathcal{O}} t' \text{ and} \\
&\qquad \text{for all } \Sigma\text{-roles } s : ((\mathcal{O} \models r \sqsubseteq s) \Rightarrow p \leadsto_s p')\} \\
A^{\mathcal{I}_i} &:= \{(t, p) \in \Delta^{\mathcal{I}_i} \mid A \in t\}
\end{aligned}
$$

We verify that interpretations $\mathcal{I}_1$ and $\mathcal{I}_2$ witness Condition 1.

Claim 1. For $i = 1, 2$, all $C \in \Xi$, and all $(t, p) \in \Delta^{\mathcal{I}_i}$, we have $(t, p) \in C^{\mathcal{I}_i}$ iff $C \in t$.

Proof of Claim 1. Let $i \in \{1, 2\}$. The proof is by induction on the structure of concepts in Ξ.

- The claim holds for concept names $C = A$, by definition of $\mathcal{I}_i$.
- The Boolean cases, $\neg C$ and $C \sqcap C'$, are immediate consequences of the hypothesis.
- Let $C = \exists r.D$. (Recall that r is possibly the universal role u.)

  "if": Suppose $\exists r.D \in t$. By existential saturation, there is a $p' = (T_1', T_2') \in \mathcal{S}^*$ such that (1) there exists $t' \in T_i'$ with $D \in t'$ and $t \leadsto_{r,\mathcal{O}}, t'$ and (2) if $\mathcal{O} \models r \sqsubseteq s$ for some Σ-role s, then $p \leadsto_s p'$. Note that t, t' are thus also u-equivalent, so $(t', p') \in \Delta^{\mathcal{I}_i}$. We distinguish cases:

  - If r is a role name, then by definition of $r^{\mathcal{I}_i}$, we have $((t, p), (t', p')) \in r^{\mathcal{I}_i}$. Since $D \in t'$, induction yields $(t', p') \in D^{\mathcal{I}_i}$. Overall, we get $(t, p) \in (\exists r.D)^{\mathcal{I}_i}$.
  - If $r = r_0^-$ is an inverse role, then (1) and (2) above imply (1') $t' \leadsto_{r_0,\mathcal{O}} t$ and (2') if $\mathcal{O} \models r_0 \sqsubseteq s$ for some Σ-role s, then $p \leadsto_s p'$. As before, we can then conclude that $((t', p'), (t, p)) \in r_0^{\mathcal{I}_i}$. Since $D \in t'$, induction yields $(t', p') \in D^{\mathcal{I}_i}$. Overall, we get $(t, p) \in (\exists r_0^-.D)^{\mathcal{I}_i}$.

  "only if": Suppose $(t, p) \in (\exists r.D)^{\mathcal{I}_i}$. Then, there is $(t', p') \in \Delta^{\mathcal{I}_i}$ with $((t, p), (t', p')) \in r^{\mathcal{I}_i}$ and $(t', p') \in D^{\mathcal{I}_i}$. By induction, the latter implies $D \in t'$. We distinguish cases:

  - If r is a role name, then by definition of $r^{\mathcal{I}_i}$, $t \leadsto_{r,\mathcal{O}} t'$ and thus $\exists r.D \in t$.
  - If $r = r_0^-$ is an inverse role, then by definition of $r_0^{\mathcal{I}_i}$, $t' \leadsto_{r_0,\mathcal{O}} t$. Thus, also $\exists r_0^-.D \in t$.

This finishes the proof of Claim 1. Claim 1 implies that $(s_1, (S_1, S_2)) \in C_1^{\mathcal{I}_1}$ and $(s_2, (S_1, S_2)) \in (\neg C_2)^{\mathcal{I}_2}$. Moreover, $\mathcal{I}_1$ and $\mathcal{I}_2$ are models of $\mathcal{O}$ since all realized types are realizable in models of $\mathcal{O}$ (we started with mosaics containing only such types).

Claim 2. The relation R defined by

$$
R = \{((t, p), (t', p)) \mid (t, p) \in \Delta^{\mathcal{I}_1}, (t', p) \in \Delta^{\mathcal{I}_2}\}
$$

is an $\mathcal{L}(\Sigma)$-bisimulation.

Proof of Claim 2. Clearly, R satisfies Condition [AtomC] due to Σ-concept name coherence.

For Condition [Forth], let $((t, p), (t', p)) \in R$ and $((t, p), (t_1, p_1)) \in r^{\mathcal{I}_1}$, for some Σ-role r, and let $p = (T_1, T_2)$ and $p_1 = (T_1', T_2')$. We distinguish cases:

- If r is a role name, then by definition of $r^{\mathcal{I}_1}$, we have (1) $t \rightsquigarrow_{r,\mathcal{O}} t_1$ and (2) for all Σ-roles s with $\mathcal{O} \models r \sqsubseteq s$, we have $p \rightsquigarrow_s p_1$. Since $t' \in T_2$ and $p \rightsquigarrow_r p_1$ there is some $t'' \in T_2'$ with $t' \rightsquigarrow_{r,\mathcal{O}} t''$. Thus, in particular, t'' is u-equivalent to t' (and thus to s_2), which implies $(t'', p_1) \in \Delta^{\mathcal{I}_2}$. The definition of $r^{\mathcal{I}_2}$ then implies that $((t', p), (t'', p_1)) \in r^{\mathcal{I}_2}$. It remains to note that the definition of R yields $((t_1, p_1), (t'', p_1)) \in R$.

- If $r = r_0^-$ is an inverse role, then by definition of $r_0^{\mathcal{I}_1}$, we have (1) $t_1 \rightsquigarrow_{r_0,\mathcal{O}} t$ and (2) for all Σ-roles s with $\mathcal{O} \models r_0 \sqsubseteq s$, we have $p_1 \rightsquigarrow_s p$. Since $t' \in T_2$ and $p_1 \rightsquigarrow_{r_0} p$ there is some $t'' \in T_2'$ with $t'' \rightsquigarrow_{r_0,\mathcal{O}} t'$. Thus, in particular, t'' is u-equivalent to t' (and thus to s_2), which implies $(t'', p_1) \in \Delta^{\mathcal{I}_2}$. The definition of $r_0^{\mathcal{I}_2}$ then implies that $((t'', p_1), (t', p)) \in r_0^{\mathcal{I}_2}$. It remains to note that the definition of R yields $((t_1, p_1), (t'', p_1)) \in R$.

Condition [Back] is dual.

Finally, we verify that R and R^- are surjective (which is only necessary if $\mathcal{L}$ allows for the universal role). Let $(t, (T_1, T_2)) \in \Delta^{\mathcal{I}_1}$. Then, $(S_1, S_2) \rightsquigarrow_u (T_1, T_2)$, by definition of $\Delta^{\mathcal{I}_1}$. This implies that there is a type $t' \in T_2$ which is u-equivalent to s_2 and thus $(t', (T_1, T_2)) \in \Delta^{\mathcal{I}_2}$. The definition of R implies $((t, (T_1, T_2)), (t', (T_1, T_2))) \in R$. The other direction is dual.

This finishes the proof of Claim 2. Note that $((s_1, (S_1, S_2)), (s_2, (S_1, S_2))) \in R$, by definition of R, and thus $\mathcal{I}_1, (s_1, (S_1, S_2)) \sim_{\mathcal{L},\Sigma} \mathcal{I}_2, (s_2, (S_1, S_2))$. ❑

Let T be a set of Ξ-types. Let $\mathcal{I}$ be an interpretation, and d_t, $t \in T$ a family of domain elements of $\mathcal{I}$. We say that $\mathcal{I}$ and d_t, $t \in T$ *jointly realize* T *modulo* $\mathcal{L}(\Sigma)$*-bisimulations* if, for all $t, t' \in T$, $\mathsf{tp}_\Xi(\mathcal{I}, d_t) = t$ and $\mathcal{I}, d_t \sim_{\mathcal{L},\Sigma} \mathcal{I}, d_{t'}$. We call T *jointly realizable under* $\mathcal{O}$ *modulo* $\mathcal{L}(\Sigma)$*-bisimulations* if there is a model $\mathcal{I}$ of $\mathcal{O}$ and elements d_t, $t \in T$ that realize T modulo $\mathcal{L}(\Sigma)$-bisimulations. Note that, in contrast to the notion of joint consistency, we require here a single model $\mathcal{I}$ of $\mathcal{O}$. However, joint realizability of a set T can be decided in double exponential time, similar to joint consistency.

In what follows, let Real denote the set of all sets of types T which are jointly realizable under $\mathcal{O}$ modulo $\mathcal{L}(\Sigma)$-bisimulations.

Lemma 13. *Let* $T_1, T_2 \in \mathsf{Real}$*. If* (T_1, T_2) *is eliminated in the elimination procedure, then we can compute an* $\mathcal{L}(\Sigma)$*-concept* I_{T_1,T_2} *such that*

1. *for all models* $\mathcal{I}$ *of* $\mathcal{O}$ *and elements* d_t, $t \in T_1$ *that realize* T_1 *modulo* $\mathcal{L}(\Sigma)$*-bisimulations,* $d_t \in I_{T_1,T_2}^{\mathcal{I}}$ *for some (equivalently: all)* $t \in T_1$*;*

2. *for all models* $\mathcal{I}$ *of* $\mathcal{O}$ *and elements* d_t, $t \in T_2$ *that realize* T_2 *modulo* $\mathcal{L}(\Sigma)$*-bisimulations,* $d_t \notin I_{T_1,T_2}^{\mathcal{I}}$ *for some (equivalently: all)* $t \in T_2$*.*

Moreover, a DAG representation of I_{T_1,T_2} *can be computed in time* $2^{2^{p(n)}}$ *for some polynomial* p *and* $n = ||\mathcal{O}|| + ||C_1|| + ||C_2||$*.*

Proof. We compute the I_{T_1,T_2} inductively in the order in which the (T_1, T_2) got eliminated in the elimination procedure. We distinguish cases why (T_1, T_2) got eliminated.

Suppose first that (T_1, T_2) was eliminated because of (failing) Σ-concept name coherence. Since T_1, T_2 are both jointly realizable, there are two cases:

(a) There is a concept name $A \in \Sigma$ such that $A \in t$ for all $t \in T_1$, but $A \notin t$, for all $t \in T_2$. Then $I_{T_1,T_2} = A$.

(b) There is a concept name $A \in \Sigma$ such that $A \in t$ for all $t \in T_2$, but $A \notin t$, for all $t \in T_1$. Then $I_{T_1,T_2} = \neg A$.

Clearly, in both cases, I_{T_1,T_2} satisfies Points 1 and 2 of Lemma 13.

Now, suppose that (T_1, T_2) was eliminated due to (failing) existential saturation from $\mathcal{S}_i$ during the elimination procedure. Since T_1, T_2 are both jointly realizable under $\mathcal{O}$, there are two cases:

(a) There exist $t \in T_1$, $\exists r.C \in t$, and a Σ-role s with $\mathcal{O} \models r \sqsubseteq s$, such that there is no $(T_1', T_2') \in \mathcal{S}_i$ such that *(i)* $(T_1, T_2) \rightsquigarrow_s (T_1', T_2')$ and *(ii)* there is $t' \in T_1'$ with $C \in t'$ and $t \rightsquigarrow_{r,\mathcal{O}} t'$. Then, take

$$I_{T_1,T_2} = \exists s.(\bigsqcup_{\substack{T_1' \in \mathsf{Real}, \\ T_1 \rightsquigarrow_s T_1', t \rightsquigarrow_{r,\mathcal{O}} t', C \in t' \in T_1'}} \bigsqcap_{\substack{T_2' \in \mathsf{Real}, \\ T_2 \rightsquigarrow_s T_2'}} I_{T_1',T_2'})$$

(b) There exist $t \in T_2$, $\exists r.C \in t$, and a Σ-role s with $\mathcal{O} \models r \sqsubseteq s$, such that there is no $(T_1', T_2') \in \mathcal{S}$ such that *(i)* $(T_1, T_2) \rightsquigarrow_s (T_1', T_2')$ and *(ii)* there is $t' \in T_2'$ with $C \in t'$ and $t \rightsquigarrow_{r,\mathcal{O}} t'$. Then, take

$$I_{T_1,T_2} = \forall s.(\bigsqcup_{\substack{T_1' \in \mathsf{Real}, \\ T_1 \rightsquigarrow_s T_1'}} \bigsqcap_{\substack{T_2' \in \mathsf{Real}, \\ T_2 \rightsquigarrow_s T_2', t \rightsquigarrow_{r,\mathcal{O}} t', C \in t' \in T_2'}} I_{T_1',T_2'})$$

We show Points 1 and 2 of the lemma for Case (a); Case (b) is dual. So suppose Case (a) applies and fix $t \in T_1, \exists r.C \in t$, and a Σ-role s witnessing that.

To show Point 1 of the lemma, let $\mathcal{I}$ be a model of $\mathcal{O}$ and fix $d_{t_1}, t_1 \in T_1$ such that $\mathcal{I}$ and the d_{t_1} realize T_1 modulo $\mathcal{L}(\Sigma)$-bisimulations. It suffices to show that $d_t \in I^{\mathcal{I}}_{T_1,T_2}$ for the type t that was fixed in the application of Case (a). Since d_t realizes t and $\exists r.C \in t$, there is some $e \in C^{\mathcal{I}}$ with $(d_t, e) \in r^{\mathcal{I}}$. Since $\mathcal{O} \models r \sqsubseteq s$, also $(d_t, e) \in s^{\mathcal{I}}$. Since the $d_{t_1}, t_1 \in T_1$ are mutually $\mathcal{L}(\Sigma)$-bisimilar and s is a Σ-role, we find elements $e_{t_1}, t_1 \in T_1$ such that:

- $e_{t_1}, t_1 \in T_1$ are mutually $\mathcal{L}(\Sigma)$-bisimilar,
- $(d_{t_1}, e_{t_1}) \in s^{\mathcal{I}}$, for all $t_1 \in T_1$,
- $e_t = e$.

Let

$$T_1' = \{\mathsf{tp}_\Xi(\mathcal{I}, e_{t_1}) \mid t_1 \in T_1\},$$

and let further $T_2' \in \mathsf{Real}$ be arbitrary with $T_2 \rightsquigarrow_s T_2'$. By definition of T_1', we have $T_1' \in \mathsf{Real}$ and $T_1 \rightsquigarrow_s T_1'$. Thus, (T_1', T_2') has been eliminated before (T_1, T_2): otherwise, Case (a) would not apply to the fixed $t, \exists r.C, s$. By induction, we can conclude that $e = e_t \in I^{\mathcal{I}}_{T_1',T_2'}$, and hence $d \in I^{\mathcal{I}}_{T_1,T_2}$.

To show Point 2 of the lemma, let $\mathcal{I}$ be a model of $\mathcal{O}$ and fix $d_{t_2}, t_2 \in T_2$ such that $\mathcal{I}$ and the d_{t_2} realize T_2 modulo $\mathcal{L}(\Sigma)$-bisimulations. Suppose, to the contrary of what has to be shown, that $d_{\widehat{t}} \in I^{\mathcal{I}}_{T_1,T_2}$ for some $\widehat{t} \in T_2$. Then, there is an e with $(d_{\widehat{t}}, e) \in s^{\mathcal{I}}$ and a $T_1' \in \mathsf{Real}$ with $T_1 \rightsquigarrow_s T_1'$ and a type $t_1' \in T_1$ with $t \rightsquigarrow_{r,\mathcal{O}} t_1'$ and $C \in t_1'$ such that

$(*)$ $e \in I^{\mathcal{I}}_{T_1',T}$ for all $T \in \mathsf{Real}$ with $T_2 \rightsquigarrow_s T$.

Since $\mathcal{I}$ and $d_{t_2}, t_2 \in T_2$ realize T_2 modulo $\mathcal{L}(\Sigma)$-bisimulations and s is a Σ-role, there are elements $e_{t_2}, t_2 \in T_2$ such that:

- $e_{t_2}, t_2 \in T_2$ are mutually $\mathcal{L}(\Sigma)$-bisimilar,
- $(d_{t_2}, e_{t_2}) \in s^{\mathcal{I}}$, for all $t_2 \in T_2$,
- $e_{\widehat{t}} = e$.

Let

$$T_2' = \{\mathsf{tp}_{\Xi}(\mathcal{I}, e_{t_2}) \mid t_2 \in T_2\}.$$

By definition of T_2', we have $T_2' \in \mathsf{Real}$ and $T_2 \rightsquigarrow_s T_2'$. Thus, (T_1', T_2') has been eliminated before (T_1, T_2): otherwise, Case (a) would not apply to the fixed $t, \exists r.C, s$. By induction, we obtain $e = e_{\widehat{t}} \notin I^{\mathcal{I}}_{T_1',T_2'}$, in contradiction to $(*)$.

For the analysis of the DAG representation, observe that we can use a single node for every I_{T_1,T_2}. Moreover, I_{T_1,T_2} looks as follows:

- If (T_1, T_2) was eliminated due to failing Σ-concept name coherence, I_{T_1,T_2} is a single concept name A or its negation $\neg A$.
- Otherwise, it is a node labeled with $\exists s$ (resp., $\forall s$), which has a single successor labeled with $\bigsqcup$. This successor has then at most double exponentially many successor nodes, each labeled with $\bigsqcap$ and each having at most double exponentially many successor nodes I_{T_1,T_2}.

Overall, we obtain double exponentially many nodes in the DAG and the DAG can be constructed in double exponential time (both in $p(||\mathcal{O}|| + ||C_1|| + ||C_2||)$). ❏

Lemma 14. *Suppose the result $\mathcal{S}^*$ of the elimination procedure does not contain a pair $(T_1, T_2) \in \mathsf{Real} \times \mathsf{Real}$ such that $C_1 \in t_1$ and $\neg C_2 \in t_2$ for some types $t_1 \in T_1$ and $t_2 \in T_2$. Then,*

$$C = \bigsqcup_{\substack{T_1 \in \mathsf{Real}: \\ \text{there is } t_1 \in T_1 \text{ with } C_1 \in t_1}} \quad \bigsqcap_{\substack{T_2 \in \mathsf{Real}: \\ \text{there is } t_2 \in T_2 \text{ with } \neg C_2 \in t_2}} I_{T_1,T_2}$$

is an $\mathcal{L}(\Sigma)$-interpolant for $C_1 \sqsubseteq C_2$ under $\mathcal{O}$. Moreover, a DAG representation of C can be computed in time $2^{2^{p(n)}}$, for some polynomial p and $n = ||\mathcal{O}|| + ||C_1|| + ||C_2||$.

Proof. We have to show that $\mathcal{O} \models C_1 \sqsubseteq C$ and $\mathcal{O} \models C \sqsubseteq C_2$.

For $\mathcal{O} \models C_1 \sqsubseteq C$, let $\mathcal{I}$ be a model of $\mathcal{O}$ and suppose $d \in C_1^{\mathcal{I}}$. Let $T_1 = \{\mathsf{tp}_\Xi(\mathcal{I}, d)\}$ consist of the single type of d. Clearly, $T_1 \in \mathsf{Real}$. Let $T_2 \in \mathsf{Real}$ be arbitrary such that $\neg C_2 \in t$, for some $t \in T_2$. By assumption of Lemma 14, (T_1, T_2) got eliminated in the elimination procedure. Point 1 of Lemma 13 implies $d \in I^{\mathcal{I}}_{T_1,T_2}$. Hence, $d \in C^{\mathcal{I}}$.

For $\mathcal{O} \models C \sqsubseteq C_2$, let $\mathcal{I}$ be a model of $\mathcal{O}$ and let $d \in (\neg C_2)^{\mathcal{I}}$. Now, let $T_1 \in \mathsf{Real}$ be arbitrary such that $C_1 \in t$ for some $t \in T_1$, and set $T_2 = \{\mathsf{tp}_\Xi(\mathcal{I}, d)\}$. Clearly, $T_2 \in \mathsf{Real}$. By assumption of Lemma 14, (T_1, T_2) got eliminated in the elimination procedure. Point 2 of Lemma 13 implies $d \notin I^{\mathcal{I}}_{T_1,T_2}$. Hence, $d \notin C^{\mathcal{I}}$.

For the analysis of the DAG representation of C, it suffices to recall that the DAG representations of the I_{T_1,T_2} provided in Lemma 13 can be computed in time $2^{2^{p(n)}}$, and to observe that C adds only one $\bigsqcup$ node and at most double exponentially many $\bigsqcap$-nodes. ❏

To conclude the section, we give some intuition as to why the proof of Theorem 11 cannot be easily adapted to logics from $\mathsf{DL}_{\mathsf{nr}}$ that allow for nominals. Observe that in any two interpretations $\mathcal{I}_1, \mathcal{I}_2$, every nominal a is realized (modulo bisimulation) in exactly one mosaic. Thus, for the mosaic elimination procedure to work (in the sense of Lemma 12) one has to "guess" for every nominal a exactly one mosaic that describes a and remove all other mosaics that mention a from the initial set $\mathcal{S}_0$ of mosaics [13]. Then, there is an interpolant for $C_1 \sqsubseteq C_2$ under $\mathcal{O}$ iff the runs of the mosaic elimination procedure for all possible guesses of the nominal mosaics in $\mathcal{S}_0$ lead to an $\mathcal{S}^*$ which does not satisfy Condition 2 of Lemma 12. It is, however, unclear how to combine these different runs in proving analogues of Lemmas 13 and 14.

7. Conclusion and Future Work

We have determined tight complexity bounds for the problem of deciding the existence of interpolants and explicit definitions in standard DLs with nominals and/or role inclusions, with and without ontologies. It would be of interest to investigate how robust our results are. For example, what happens for extensions of the DLs considered here with number restrictions?

We have also performed first steps in the analysis of the computation problem, but many interesting problems remain to be addressed. First, note that our analysis only applies to the case with ontologies and we expect interpolants in the ontology free case to be one exponential smaller (if the universal role is not present). Second, we have provided only upper bounds on the size of interpolants and it remains to see whether the construction is optimal (we conjecture it to be). Finally, it is of great interest to compute interpolants also in the presence of nominals. An alternative approach might be to derive them from a suitably constrained proof of $\mathcal{O} \models C_1 \sqsubseteq C_2$ in a suitable proof system, see e.g. [20].

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