# On non-local propositional and one-variable quantified $\mathsf{CTL}^*$


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## 1 Introduction

This paper continues the investigation of the computational behaviour of first-order branching time temporal logics started in [6]. A 'negative' result obtained in [6] is the undecidability of the one-variable fragment of quantified computational tree logic $\mathsf{CTL}^*$ (both bundled [1, 9] and 'unbundled' versions, and even with sole temporal operator 'some time in the future'). On the other hand, it was shown that by restricting applications of first-order quantifiers to *state* (i.e., path-independent) formulas, and applications of temporal operators and path quantifiers to formulas with at most one free variable, decidable fragments can be obtained.

Here we prove decidability of another kind of fragment of first-order $\mathsf{CTL}^*$, the so-called *weak one-variable fragment*, in which quantifiers are not restricted to state formulas, but only the next-time operator may be applied to open formulas, while all other temporal operators and path quantifiers are applicable only to sentences. To obtain this result, we first show decidability of the non-local version of propositional $\mathsf{CTL}^*$, where truth values of atoms may depend on the branch of evaluation,[1] and then reduce the weak one-variable fragment to this logic. The main technical instrument in both proofs is the method of quasimodels [4].

## 2 Decidability of non-local $\mathsf{PCTL}^*$

The propositional language $\mathcal{PCTL}^*$ [3, 7] extends propositional logic with temporal connectives $\mathsf{U}, \mathsf{S}$ ('until,' 'since') and a path quantifier $\mathsf{E}$ ('there exists a branch (or history)'). The dual path quantifier $\mathsf{A}$ ('for all branches (or histories)') is defined as an abbreviation: $\mathsf{A}\varphi = \neg\mathsf{E}\neg\varphi$. Other standard abbreviations we need are: $\Diamond_F\varphi = \top\,\mathsf{U}\,\varphi$, $\Box_F\varphi = \neg\Diamond_F\neg\varphi$, $\Diamond_P\varphi = \top\,\mathsf{S}\,\varphi$, $\bigcirc\varphi = \bot\,\mathsf{U}\,\varphi$, $\bigcirc_P\varphi = \bot\,\mathsf{S}\,\varphi$ ('some time in the future', 'always in the future,' 'some time in the past,' 'at the next moment' and 'at the previous moment').

This language is interpreted in (bundled and unbundled) models based on $\omega$-trees. A *tree* is a strict partial order $\mathfrak{T} = \langle T, <\rangle$ such that for all $w \in T$, the set $\{v \in T : v < w\}$ is well-ordered by $<$. When we write $\mathfrak{T}$ for a tree, it will be implicit that $\mathfrak{T} = \langle T, <\rangle$. For $t \in T$, let $ht(t) = |\{u \in T : u < t\}|$. A *full*

[1]This contrasts with the behaviour of process logic, the local version of which is decidable, while the non-local one is undecidable [2].

*branch* of $\mathfrak{F}$ is a maximal linearly-ordered subset of $T$. An $\omega$-*tree* is a rooted tree whose full branches, ordered by $<$, are all order-isomorphic to the natural numbers $\langle \mathbb{N}, < \rangle$. A *bundle* on $\mathfrak{T}$ is a set $\mathcal{H}$ of full branches of $\mathfrak{T}$ with $\bigcup \mathcal{H} = T$.

In this paper we deal with the 'non-local' variant of $\mathcal{PCTL}^*$ in which truth values of atoms can depend on the branch of evaluation. Thus, a *bundled model* has the form $\mathfrak{M} = \langle \mathfrak{T}, \mathcal{H}, h \rangle$, where $\mathfrak{T}$ is an $\omega$-tree, $\mathcal{H}$ is a bundle on $\mathfrak{T}$, and $h : \{\text{atoms}\} \to \wp(\{(\beta,t) : t \in \beta \in \mathcal{H}\})$. $\mathfrak{M}$ is a *full tree model* if $\mathcal{H}$ is the set of all full branches of $\mathfrak{T}$. The semantics of $\mathcal{PCTL}^*$ is now defined as follows, where $t \in \beta \in \mathcal{H}$:

- for an atom $p$, $(\mathfrak{M}, \beta, t) \models p$ iff $(\beta, t) \in h(p)$;
- the booleans are defined as usual;
- $(\mathfrak{M}, \beta, t) \models \varphi \mathsf{U} \psi$ iff there is $u > t$ with $u \in \beta$, $(\mathfrak{M}, \beta, u) \models \psi$, and $(\mathfrak{M}, \beta, v) \models \varphi$ for all $v \in (t,u)$, where $(t,u) = \{v \in T : t < v < u\}$;
- $(\mathfrak{M}, \beta, t) \models \varphi \mathsf{S} \psi$ iff there is $u < t$ with $(\mathfrak{M}, \beta, u) \models \psi$ and $(\mathfrak{M}, \beta, v) \models \varphi$ for all $v \in (u,t)$;
- $(\mathfrak{M}, \beta, t) \models \mathsf{E}\varphi$ iff $(\mathfrak{M}, \gamma, t) \models \varphi$ for some $\gamma \in \mathcal{H}$ with $t \in \gamma$.

LEMMA 1. *If a $\mathcal{PCTL}^*$-formula $\varphi$ is satisfiable in a full (bundled) tree model, then $\varphi$ is satisfiable in a full (respectively, bundled) tree model based on a countable $\omega$-tree.*

*Proof.* Let $\mathfrak{M} = \langle \mathfrak{T}, \mathcal{H}, h \rangle$ be a tree model. We may view $\mathfrak{M}$ as a two-sorted first-order structure, the two sorts being $\mathfrak{T}$ and $\mathcal{H}$. Taking a countable elementary substructure of this yields a *bundled* tree model $\mathfrak{N} = \langle \mathfrak{T}_0, \mathcal{H}_0, h_0 \rangle$ whose tree $\mathfrak{T}_0$ and bundle $\mathcal{H}_0$ are countable. Here, $h_0(p) = h(p) \cap \{(\beta,t) : t \in \beta \in \mathcal{H}_0\}$ for any atom $p$. It is easy to translate $\mathcal{PCTL}^*$-formulas to two-sorted first-order formulas with the same meaning. It follows that for any $\beta \in \mathcal{H}_0$, $t \in \beta$, and any formula $\psi$, we have $(\mathfrak{M}, \beta, t) \models \psi$ iff $(\mathfrak{N}, \beta, t) \models \psi$. This completes the proof for the bundled case.

Suppose now that $\mathcal{H}$ contains all full branches of $\mathfrak{T}$ and let $\overline{\mathfrak{N}} = \left\langle \mathfrak{T}_0, \overline{\mathcal{H}}_0, \overline{h}_0 \right\rangle$ be the full tree model based on $\mathfrak{N}$, where $\overline{\mathcal{H}}_0 \supseteq \mathcal{H}_0$ is the set of all full branches of $\mathfrak{T}_0$, and $\overline{h}_0(p) = h(p) \cap \{(t,\beta) : t \in \beta \in \overline{\mathcal{H}}_0\}$, for an atom $p$. We claim that for all $\mathcal{PCTL}^*$-formulas $\psi$, all full branches $\gamma$ of $\mathfrak{T}_0$ and all $t \in \gamma$, we have

$$(\mathfrak{M}, \gamma, t) \models \psi \quad \text{iff} \quad (\overline{\mathfrak{N}}, \gamma, t) \models \psi.$$

The proof is by induction on $\psi$. The atomic, boolean, and temporal cases are trivial. Consider the case $\mathsf{E}\psi$ and inductively assume the result for $\psi$. If $(\mathfrak{M}, \gamma, t) \models \mathsf{E}\psi$, pick $\beta \in \mathcal{H}_0$ containing $t$. Clearly, $(\mathfrak{M}, \beta, t) \models \mathsf{E}\psi$, so $(\mathfrak{N}, \beta, t) \models \mathsf{E}\psi$. So there is $\beta' \in \mathcal{H}_0$ with $(\mathfrak{N}, \beta', t) \models \psi$. Thus, $(\mathfrak{M}, \beta', t) \models \psi$. Inductively, $(\overline{\mathfrak{N}}, \beta', t) \models \psi$. So $(\overline{\mathfrak{N}}, \gamma, t) \models \mathsf{E}\psi$, as required. The converse implication is easy. $\square$

Fix a $\mathcal{PCTL}^*$-formula $\varphi$.

DEFINITION 2. $sub(\varphi)$ will denote the set of subformulas of $\varphi$ and their negations. A *type* for $\varphi$ is a subset $\sigma$ of $sub(\varphi)$ such that $\psi \wedge \chi \in \sigma$ iff $\psi \in \sigma$ and $\chi \in \sigma$, for every $\psi \wedge \chi \in sub(\varphi)$, and $\neg\psi \in \sigma$ iff $\psi \notin \sigma$, for every $\neg\psi \in sub(\varphi)$. A set $\Sigma$ of types is said to be *coherent* if it is non-empty and for all $\mathsf{E}\psi \in sub(\varphi)$, the conditions $\mathsf{E}\psi \in \bigcap \Sigma$, $\mathsf{E}\psi \in \bigcup \Sigma$, and $\psi \in \bigcup \Sigma$ are equivalent.

Fix an $\omega$-tree $\mathfrak{T} = \langle T, < \rangle$.

DEFINITION 3. Given a non-empty set $\Sigma_t$ of types for each $t \in T$, and a full branch $\beta$ of $\mathfrak{T}$, a *run in* $\beta$ is a map $r : \beta \to \bigcup_{t\in\beta} \Sigma_t$ such that

- $r(t) \in \Sigma_t$ for each $t \in \beta$,
- for all $\psi \mathsf{U} \chi \in sub(\varphi)$ and all $t \in \beta$, we have $\psi \mathsf{U} \chi \in r(t)$ iff there is $u > t$ with $u \in \beta$, $\chi \in r(u)$, and $\psi \in r(v)$ for all $v \in (t,u)$,
- for all $\psi \mathsf{S} \chi \in \mathcal{S}(\varphi)$ and all $t \in \beta$, we have $\psi \mathsf{S} \chi \in r(t)$ iff there is $u < t$ with $\chi \in r(u)$ and $\psi \in r(v)$ for all $v \in (u,t)$.

DEFINITION 4. A family $(\Sigma_t : t \in T)$ of coherent sets of types is said to be an *unbundled quasimodel for* $\varphi$ *over* $\mathfrak{T}$ if

1. $\varphi \in \sigma$ for some $t \in T$ and $\sigma \in \Sigma_t$,
2. for all $t \in T$, $\sigma \in \Sigma_t$, there is a full branch $\beta$ of $\mathfrak{T}$ containing $t$ and a run $r$ in $\beta$ such that $r(t) = \sigma$,
3. for each full branch $\beta$ of $\mathfrak{T}$, there exists a run in $\beta$.

$(\Sigma_t : t \in T)$ is a *bundled quasimodel* for $\varphi$ over $\mathfrak{T}$ if it satisfies conditions 1 and 2.

LEMMA 5. $\varphi$ *is satisfied in a (bundled) model iff there is a (bundled) quasimodel for* $\varphi$ *over a countable* $\omega$*-tree.*

*Proof.* Let $\mathfrak{M} = \langle \mathfrak{T}, \mathcal{H}, h \rangle$ be such that $(\mathfrak{M}, \beta_0, t_0) \models \varphi$ for some $\beta_0 \in \mathcal{H}$ and $t_0 \in \beta_0$. By Lemma 1, we can assume that $\mathfrak{T}$ is countable. For $\beta \in \mathcal{H}$ and $t \in \beta$, let $\mathrm{tp}(t,\beta) = \{\psi \in sub(\varphi) : (\mathfrak{M}, \beta, t) \models \psi\}$. Clearly, $\mathrm{tp}(t,\beta)$ is a type for $\varphi$. For $t \in T$, let $\Sigma_t = \{\mathrm{tp}(t,\beta) : t \in \beta \in \mathcal{H}\}$. Clearly, $\Sigma_t$ is coherent. For any $\beta \in \mathcal{H}$, the map $r_\beta : t \mapsto \mathrm{tp}(t,\beta)$ is then a run in $\beta$. We claim that $(\Sigma_t : t \in T)$ is a quasimodel for $\varphi$ over $\mathfrak{T}$ (a bundled one if $\mathfrak{M}$ is bundled, and an unbundled one otherwise). As $(\mathfrak{M}, \beta_0, t_0) \models \varphi$, we have $\varphi \in \mathrm{tp}(t_0, \beta_0) \in \Sigma_{t_0}$. For each $t \in T$ and $\sigma \in \Sigma_t$, we have $\sigma = \mathrm{tp}(t,\beta)$ for some $\beta \in \mathcal{H}$ containing $t$, so $r_\beta(t) = \sigma$ and condition 2 of Definition 4 holds. Finally, for all $\beta \in \mathcal{H}$, $r_\beta$ is a run in $\beta$, so if $\mathfrak{M}$ is a full tree model, it is clear that condition 3 holds.

Conversely, let $(\Sigma_t : t \in T)$ be a quasimodel for $\varphi$ over a countable $\omega$-tree $\mathfrak{T}$. By replacing $\mathfrak{T}$ by $\mathfrak{T} \otimes {}^{<\omega}2 =_{\mathrm{def}} \{(t,\eta) : t \in T, \eta \in {}^{<\omega}2, ht(t) = |\eta|\}$ — a countable $\omega$-tree when ordered by $(t,\eta) < (u,\xi)$ iff $t < u$ in $T$ and $\eta$ is an initial segment of $\xi$ — and letting $\Sigma_{(t,\eta)} = \Sigma_t$ for all $t,\eta$, we can assume that $(*)$ for each $t \in T$ and $\sigma \in \Sigma_t$, there are $2^\omega$ full branches $\beta$ of $\mathfrak{T}$ containing $t$ such that there is a run $r$ in $\beta$ with $r(t) = \sigma$.

Each $\Sigma_t$ is finite, so there are countably many pairs $(t,\sigma)$ with $t \in T$, $\sigma \in \Sigma_t$. Enumerate them as $(t_n, \sigma_n)$ $(n < \omega)$. Inductively, using $(*)$, choose a full branch $\beta_n \ni t_n$ for each $n < \omega$, such that (i) there is a run $r_{\beta_n}$ in $\beta_n$ with $r_{\beta_n}(t_n) = \sigma_n$, and (ii) $\beta_n \neq \beta_m$ for all $m < n$. If we have a bundled quasimodel, let $\mathcal{H} = \{\beta_n : n < \omega\}$. This is clearly a bundle on $\mathfrak{T}$. If we have an unbundled quasimodel, let $\mathcal{H}$ be the set of all full branches of $\mathfrak{T}$, and further choose for each $\beta \in \mathcal{H} \setminus \{\beta_n : n < \omega\}$ a run $r_\beta$ in $\beta$; this can be done by condition 3 of Definition 4. So we have defined a run $r_\beta$ in $\beta$, for each $\beta \in \mathcal{H}$.

Now define a model $\mathfrak{M} = \langle \mathfrak{T}, \mathcal{H}, h \rangle$ where $h(p) = \{(\beta,t) : t \in \beta \in \mathcal{H},\ p \in r_\beta(t)\}$ for each atom $p$.

CLAIM. For all $\beta \in \mathcal{H}$, all $t \in \beta$, and all $\psi \in sub(\varphi)$, we have $(\mathfrak{M}, \beta, t) \models \psi$ iff $\psi \in r_\beta(t)$.
PROOF OF CLAIM. The proof is by induction on $\psi$. For atomic $\psi = p$, we have $(\mathfrak{M}, \beta, t) \models p$ iff $(\beta, t) \in h(p)$, iff $p \in r_\beta(t)$ as required. The boolean cases are trivial. For $\psi \mathsf{U} \chi \in sub(\varphi)$, we have $(\mathfrak{M}, \beta, t) \models \psi \mathsf{U} \chi$ iff there is $u \in \beta$ with $u > t$, $(\mathfrak{M}, \beta, u) \models \chi$, and $(\mathfrak{M}, \beta, v) \models \psi$ for all $v \in (t,u)$.

Inductively, this holds iff there is $u \in \beta$ with $u > t$, $\chi \in r_\beta(u)$, and $\psi \in r_\beta(v)$ for all $v \in (t,u)$. Since $r_\beta$ is a run in $\beta$, this is iff $\psi \mathsf{U} \chi \in r_\beta(t)$, as required. The case of $\mathsf{S}$ is similar.

Finally, for $\mathsf{E}\psi \in sub(\varphi)$, we have $(\mathfrak{M},\beta,t) \models \mathsf{E}\psi$ iff $(\mathfrak{M},\gamma,t) \models \psi$ for some $\gamma \in \mathcal{H}$ with $t \in \gamma$. Inductively, this is iff $\psi \in r_\gamma(t)$ for some $\gamma \in \mathcal{H}$ with $t \in \gamma$. But evidently, $\Sigma_t = \{r_\gamma(t) : \gamma \in \mathcal{H},\ t \in \gamma\}$, so this is iff $\psi \in \bigcup \Sigma_t$. Since $\Sigma_t$ is coherent, this is iff $\mathsf{E}\psi \in r_\beta(t)$, as required. The claim is proved.

Now let $t \in T$ be such that $\varphi \in \sigma$ for some $\sigma \in \Sigma_t$. We may choose $n < \omega$ with $(t,\sigma) = (t_n,\sigma_n)$. Then $t \in \beta_n \in \mathcal{H}$ and $r_{\beta_n}(t) = \sigma$, so by the claim, $(\mathfrak{M},\beta_n,t) \models \varphi$. Thus, $\varphi$ has a model. $\square$

LEMMA 6. *Given a $\mathcal{PCTL}^*$-formula $\varphi$, it is decidable whether $\varphi$ has an unbundled quasimodel over a countable $\omega$-tree. The same holds for bundled quasimodels.*

*Proof.* Given $\varphi$, we can effectively construct the set $\mathcal{C}$ of all coherent sets of types. A quasimodel over an $\omega$-tree $\mathfrak{T}$ has the form $(\Sigma_t : t \in T)$ where $\Sigma_t \in \mathcal{C}$ for each $t$; we will express this by unary relation variables $P_\Sigma$ for each $\Sigma \in \mathcal{C}$, the aim being that $P_\Sigma$ is true at $t$ iff $\Sigma_t = \Sigma$. We then express the stipulations of Definition 4 in terms of the $P_\Sigma$, as follows. Let $R_\psi$ ($\psi \in sub(\varphi)$) be unary relation variables. For a type $\sigma$ for $\varphi$, let

$$\chi_\sigma(x) \quad = \quad \bigwedge_{\psi \in \sigma} R_\psi(x) \ \wedge \bigwedge_{\psi \in sub(\varphi)\setminus\sigma} \neg R_\psi(x).$$

The formula $\chi_\sigma(x)$ says that the $R_\psi(x)$ define the type $\sigma$ at $x$. For a unary relation variable $B$, let $\rho(B)$ be the conjunction of:

- $\bigwedge_{\Sigma \in \mathcal{C}} \forall x \big(B(x) \wedge P_\Sigma(x) \to \bigvee_{\sigma \in \Sigma} \chi_\sigma(x)\big)$
- $\forall x \big(R_{\psi_1 \mathsf{U} \psi_2}(x) \leftrightarrow \exists y (B(y) \wedge x < y \wedge R_{\psi_2}(y) \wedge \forall z (x < z < y \to R_{\psi_1}(z)))\big)$, for all $\psi_1 \mathsf{U} \psi_2 \in sub(\varphi)$;
- $\forall x \big(R_{\psi_1 \mathsf{S} \psi_2}(x) \leftrightarrow \exists y (y < x \wedge R_{\psi_2}(y) \wedge \forall z (y < z < x \to R_{\psi_1}(z)))\big)$, for all $\psi_1 \mathsf{S} \psi_2 \in sub(\varphi)$.

So assuming that $B$ defines a full branch, $\rho(B)$ says that the $R_\psi$ define a run in $B$. Let $\beta(B)$ be a monadic second-order formula expressing that $B$ is a full branch (a maximal linearly-ordered set). Thus, the following formula $\mu$ is effectively constructible from $\varphi$:

$$\begin{aligned}\exists_{\Sigma \in \mathcal{C}} P_\Sigma \Bigg( \forall x \bigvee_{\Sigma \in \mathcal{C}} \Big[P_\Sigma(x) \wedge \bigwedge_{\substack{\Sigma' \in \mathcal{C} \\ \Sigma \neq \Sigma'}} \neg P_{\Sigma'}(x)\Big] \ \wedge\ \exists x \bigvee_{\substack{\Sigma \in \mathcal{C} \\ \varphi \in \bigcup \Sigma}} P_\Sigma(x) \quad \wedge\ \forall B \Big[\beta(B) \to \exists_{\psi \in sub(\varphi)} R_\psi\ \rho(B)\Big] \\ \wedge\ \forall x \bigwedge_{\substack{\Sigma \in \mathcal{C} \\ \sigma \in \Sigma}} \Big[P_\Sigma(x) \to \exists B \Big(\beta(B) \wedge B(x) \wedge \exists_{\psi \in sub(\varphi)} R_\psi (\rho(B) \wedge \chi_\sigma(x))\Big)\Big] \Bigg).\end{aligned}$$

(If we are dealing with bundled models, we omit the conjunct $\forall B[\beta(B) \to \exists_{\psi \in sub(\varphi)} R_\psi\ \rho(B)]$.) It should be clear that for any $\omega$-tree $\mathfrak{T}$, we have $\mathfrak{T} \models \mu$ iff there is a quasimodel for $\varphi$ over $\mathfrak{T}$ (bundled or unbundled, as appropriate). It follows from decidability of S2S [8] that it is decidable whether a given monadic second-order sentence is true in some countable $\omega$-tree. The lemma now follows. $\square$

As a consequence of Lemmas 5 and 6 we finally obtain

THEOREM 7. *It is decidable whether a $\mathcal{PCTL}^*$-formula has a full tree model in the non-local semantics. The same holds for bundled models.*

## 3 Decidability of the weak one-variable fragment of $\mathcal{QPCTL}^*$

Fix an individual variable $x$ and denote by $\mathcal{QPCTL}_1^*$ the one-variable fragment of first-order $\mathcal{PCTL}^*$, which can be defined as the closure of the sets $\{P_0(x), P_1(x), \dots\}$ of unary predicates and $\{p_0, p_1, \dots\}$ of propositional variables under the operators $\exists x$, $\wedge$, $\neg$, $\mathsf{E}$, $\bigcirc$, $\mathsf{U}$ and $\mathsf{S}$. Note that now $\bigcirc$ is regarded as a primitive operator. The *weak one-variable fragment* $\mathcal{QPCTL}_w^*$ of $\mathcal{QPCTL}^*$ consists of all $\mathcal{QPCTL}_1^*$-formulas in which the temporal operators $\mathsf{U}$ and $\mathsf{S}$ and the path-quantifier $\mathsf{E}$ are applied to sentences only. Thus, $\bigcirc$ is the only temporal operator which can be applied to open formulas.

A $\mathcal{QPCTL}_1^*$-*model* is a quadruple of the form $\mathfrak{M} = \langle \mathfrak{T}, \mathcal{H}, D, I \rangle$, where $\mathfrak{T} = \langle T, < \rangle$ is an $\omega$-tree, $\mathcal{H}$ is a bundle on $\mathfrak{T}$, $D$ is a non-empty set, the *domain* of $\mathfrak{M}$, and $I$ is a function associating with every time point $w \in T$ a usual first-order structure

$$I(w) = \left\langle D, P_0^{I(w)}, \dots, p_0^{I(w)}, \dots \right\rangle$$

in the signature of $\mathcal{QPCTL}_1^*$—the *state* of $\mathfrak{M}$ at $w$. As before, $\mathfrak{M}$ is called a *full* model if $\mathcal{H}$ contains all full branches of $\mathfrak{T}$. An *assignment* in $\mathfrak{M}$ is a function $\mathfrak{a} : \{x\} \to D$. Let $w \in \beta \in \mathcal{H}$ and let $\varphi$ be a formula. The *truth-relation* $(\mathfrak{M}, \beta, w) \models^{\mathfrak{a}} \varphi$ (or $(\beta, w) \models^{\mathfrak{a}} \varphi$ if $\mathfrak{M}$ is understood, or $(\mathfrak{M}, \beta, w) \models \varphi[d]$ where $\mathfrak{a}(x) = d$) is defined inductively by:

- $(\beta, w) \models^{\mathfrak{a}} \alpha$ iff $I(w) \models^{\mathfrak{a}} \alpha$, for atomic $\alpha$;
- $(\beta, w) \models^{\mathfrak{a}} \exists x \psi$ iff $(\beta, w) \models^{\mathfrak{b}} \psi$ for some assignment $\mathfrak{b}$;

for the temporal operators and path quantifiers, the definition is the same as in the propositional case. Note that we have returned to the traditional '*local*' semantics in which truth values of atoms do not depend on the branch $\beta$ of evaluation.

The main result we prove in the remainder of this section is the following

THEOREM 8. *The satisfiability problem for $\mathcal{QPCTL}_w^*$-formulas in both bundled and full models is decidable.*

REMARK 9. (1) We remind the reader that the satisfiability problem for the full one variable fragment $\mathcal{QPCTL}_1^*$ in both bundled and full models is undecidable [6].

(2) Actually, using somewhat more sophisticated machinery (in particular, a mosaic technique) one can generalise Theorem 8 to the two variable, monadic, and guarded fragments of $\mathcal{QPCTL}$ in which $\bigcirc$ can be applied to formulas with at most one free variable and the other temporal operators and path quantifiers only to sentences.

(3) Theorem 8 and its generalisation above still holds if we extend $\mathcal{QPCTL}_w^*$ with individual constants; however, functional symbols and equality may lead to undecidability, cf. [4].

We will prove this result in two steps. Firstly, we show that a $\mathcal{QPCTL}_w^*$-formula is satisfiable iff it is satisfiable in certain quasimodels. Then we will reduce satisfiability in quasimodels to non-local propositional satisfiability. We begin the proof of Theorem 8 by recalling that the bundled case is reducible to the 'unbundled' one [6]. So it is enough to consider satisfiability in full models $\mathfrak{M} = \langle \mathfrak{T}, D, I \rangle$.

Fix a $\mathcal{QPCTL}_w^*$-sentence $\varphi$. For simplicity we may assume that any *subsentence* $\bigcirc\psi$ of $\varphi$ is replaced by $\bot \,\mathsf{U}\, \psi$. Thus, $\bigcirc$ is only applied to formulas with free variable $x$. We define $sub(\varphi)$ and types for $\varphi$ as in Definition 2. For every formula $\theta(x) = \bigcirc\psi(x) \in sub(\varphi)$ we reserve fresh unary predicates $P_\theta^i(x)$, and for every $\theta$ of the form $\mathsf{E}\psi$, $\psi_1 \mathsf{U} \psi_2$, or $\psi_1 \mathsf{S} \psi_2$ in $sub(\varphi)$ we reserve fresh propositional variables $p^i_\theta$, where $i = 0, 1, \ldots$. The $P^i_\theta(x)$ and $p^i_\theta$ are called the *i-surrogates* of $\theta(x)$ and $\theta$, respectively. For $\psi \in sub(\varphi)$, denote by $\psi^i$ the result of replacing in $\psi$ all its subformulas of the form $\bigcirc\psi$, $\psi \mathsf{U} \chi$, $\psi \mathsf{S} \chi$, or $\mathsf{E}\psi$ that are not within the scope of another occurrence of a non-classical operator by their $i$-surrogates. Thus, $\psi^i$ is a purely first-order (non-temporal) formula. Let $\Gamma^i = \{\chi^i : \chi \in \Gamma\}$ for any set $\Gamma \subseteq sub(\varphi)$.

The idea behind these definitions is as follows. The formulas $\chi^i$ abstract from the temporal component of $\chi$ and can be evaluated in a first-order structure without taking into account its temporal evolution. Of course, later we have to be able to reconstruct the truth value of $\chi$ in temporal models from the truth value of the $\chi^i$. In contrast to the linear time case, we need a list of abstractions $\chi^0, \chi^1, \ldots$, since the temporal evolution depends on branches. So, intuitively, each $i < \omega$ represents a branch. (Actually, we will see that finitely many $i < \omega$ are enough, since we have to represent branches only up to a certain equivalence relation).

Let $\rho(\varphi) = 4^{4^{|sub(\varphi)|}}$.

DEFINITION 10. A *state candidate* for $\varphi$ is a pair $\Theta = (\mathcal{S}, \mathcal{T})$ in which:

(i) $\mathcal{S} = \{S_1, \ldots, S_k\}$, where each $S_i$ is a set of types for $\varphi$ such that, for every sentence $\psi$, we have $\psi \in \sigma$ iff $\psi \in \sigma'$, for any $\sigma, \sigma' \in S_i$, and for every $\mathsf{E}\psi \in sub(\varphi)$,

$$\mathsf{E}\psi \in \bigcap_{i \le k} \bigcap S_i \quad \text{iff} \quad \mathsf{E}\psi \in \bigcup_{i \le k} \bigcap S_i \quad \text{iff} \quad \psi \in \bigcup_{i \le k} \bigcap S_i.$$

(ii) $\mathcal{T}$ is a set of maps $\tau : \{1, \ldots, n_\Theta\} \to \bigcup_{i \le k} S_i$, called *traces*, where $n_\Theta \le \rho(\varphi)$ is a natural number depending on $\Theta$ and such that $\big\{\{\tau(i) : \tau \in \mathcal{T}\} : i \le n_\Theta\big\} = \mathcal{S}$.

The set of sentences in $\bigcap\{\tau(i) : \tau \in \mathcal{T}\}$ will be denoted by $\Theta(i)$. For a trace $\tau$, we set

$$\overline{\tau} = \bigcup_{i \le n_\Theta} (\tau(i))^i, \qquad \overline{\mathcal{T}} = \{\overline{\tau} : \tau \in \mathcal{T}\}.$$

DEFINITION 11. Let $\Theta = \langle \mathcal{S}, \mathcal{T} \rangle$ be a state candidate for $\varphi$ and

$$\mathfrak{D} = \left\langle D, P_0^{\mathfrak{D}}, \ldots, p_0^{\mathfrak{D}}, \ldots \right\rangle$$

a first-order structure in the signature of $\mathcal{QPCTL}^*_1$. For every $a \in D$ we define the *trace* of $a$ (with respect to $\Theta$) as

$$tr(a) = \{\psi \in \bigcup_{i \le n_\Theta} (sub(\varphi))^i : \mathfrak{D} \models \psi[a]\}.$$

We say that $\mathfrak{D}$ *realises* $\Theta$ if $\overline{\mathcal{T}} = \{tr(a) : a \in D\}$.

State candidates represent states $w$ of temporal models. The intuition behind this definition will be clear from the proof of the theorem below. Here we only say that, roughly, the components $S_i$ of a state candidate $\Theta = \langle \mathcal{S}, \mathcal{T} \rangle$ represent the states of a moment $w$ in different branches, and each trace $\tau \in \mathcal{T}$ shows the types of one element of the domain of $w$ in these states (i.e., its possible states in different histories).

It follows immediately from the definition that we have:

LEMMA 12. *A state candidate $\Theta = \langle \mathcal{S}, \mathcal{T} \rangle$ for $\varphi$ is realisable iff the first-order sentence*

$$\alpha_\Theta = \bigwedge_{\tau \in \mathcal{T}} \exists x \overline{\tau} \wedge \forall x \bigvee_{\tau \in \mathcal{T}} \overline{\tau}$$

*is satisfiable.*

DEFINITION 13. A *connection* is a quadruple $(\Delta, \Theta, R, N)$ consisting of realisable state candidates $\Delta = (\mathcal{S}, \mathcal{T})$ and $\Theta = (\mathcal{U}, \mathcal{V})$, a relation $R \subseteq \mathcal{T} \times \mathcal{V}$ with domain $\mathcal{T}$ and range $\mathcal{V}$, and a relation $N \subseteq \{1, \ldots, n_\Delta\} \times \{1, \ldots, n_\Theta\}$ with range $\{1, \ldots, n_\Theta\}$, such that for all $(i, j) \in N$, all $(\tau, \tau') \in R$, and all $\bigcirc\psi \in sub(\varphi)$, we have $\bigcirc\psi \in \tau(i)$ iff $\psi \in \tau'(j)$.

A connection describes how (the abstract representation $\Theta$ of) a state $w$ is related to (the abstract representation $\Delta$ of) its immediate predecessor. To this end, the relation $R$ between the traces in both representations is fixed.

For an $\omega$-tree $\mathfrak{T} = \langle T, < \rangle$ and $w \in T$, denote by $B(w)$ the set of full branches coming through $w$.

DEFINITION 14. A *quasimodel* for $\varphi$ over $\mathfrak{T} = \langle T, < \rangle$ is a map $f$ associating with the root $w_0$ of $\mathfrak{T}$ a pair $f(w_0) = (\Theta_{w_0}, g_{w_0})$, where $\Theta_{w_0}$ is a realisable state candidate, and with every non-root $w \in T$ a pair $f(w) = (C_w, g_w)$, where $C_w = (\Delta_w, \Theta_w, R_w, N_w)$ is a connection, and all $g_w$, for $w \in T$, are functions from $B(w)$ onto $\{1, \ldots, n_{\Theta_w}\}$ such that the following hold:

1. if $v \in T$ is the immediate predecessor of $w$, then $\Theta_v = \Delta_w$ and $N_w = \{(g_v(\beta), g_w(\beta)) : \beta \in B(w)\}$;
2. for all $\beta \in B(w)$, $\chi \mathsf{U} \psi \in \Theta_w(g_w(\beta))$ iff there exists $u > w$ with $u \in \beta$, $\psi \in \Theta_u(g_u(\beta))$ and $\chi \in \Theta_v(g_v(\beta))$ for all $v \in (w, u)$ $\quad$ ($\Theta(i)$ was defined after Definition 10);
3. for all $\beta \in B(w)$, $\chi \mathsf{S} \psi \in \Theta_w(g_w(\beta))$ iff there exists $u < w$ with $\psi \in \Theta_u(g_u(\beta))$ and $\chi \in \Theta_v(g_v(\beta))$ for all $v \in (u, w)$.

We say that $f$ *satisfies* $\varphi$ if there is $w \in T$ such that $\Theta_w = (\mathcal{S}_w, \mathcal{T}_w)$ and $\varphi \in \bigcup S$ for some $S \in \mathcal{S}_w$.

While the connections take care of the truth values of 'local' formulas of the form $\bigcirc\chi$, quasimodels take care of the remaining 'global' temporal operators.

THEOREM 15. *$\varphi$ is satisfiable in a $\mathcal{QPCTL}_1^*$-model iff there exists a quasimodel satisfying $\varphi$.*

*Proof.* ($\Rightarrow$) Suppose that $\varphi$ is satisfied in some model. We may replace its tree $\mathfrak{T}$ by $\mathfrak{T}^+ = \mathfrak{T} \otimes {}^{<\omega}\omega$, as in the proof of Lemma 5; every branch of $\mathfrak{T}$ is 'duplicated' $\omega$ times in $\mathfrak{T}^+$ at each node, and $\varphi$ is still satisfied in the resulting model $\mathfrak{M} = \langle \mathfrak{T}^+, D, I \rangle$. So $(\mathfrak{M}, \sigma, v) \models^{\mathfrak{a}} \varphi$ for some $v \in T^+$, $\sigma \in B(v)$ (defined w.r.t. $\mathfrak{T}^+$) and some assignment $\mathfrak{a}$. Given $w \in T^+$ and $\beta \in B(w)$, let

$$S(\beta, w) = \{\mathrm{tp}(\beta, w, a) : a \in D\},$$

where

$$\mathrm{tp}(\beta, w, a) = \{\psi \in sub(\varphi) : (\mathfrak{M}, \beta, w) \models \psi[a]\}.$$

Let $\mathcal{S}_w = \{S(\beta, w) : \beta \in B(w)\}$. We extract from $\mathfrak{T}^+$ a subtree $\mathfrak{T}' = \langle T', <' \rangle$ in which every node has at most $\xi(\varphi) = 2^{2^{|sub(\varphi)|}}$ immediate successors. To this end, we inductively define $T'_n \subseteq T^+$ with this property. Set $T'_0 = \{w_0\}$, where $w_0$ is the root of $\mathfrak{T}^+$. Given $T'_n$, for each $w \in T'_n$ with $ht(w) = n$, and each $S \in \mathcal{S}_w$, we pick $\beta_S \in B(w)$ such that $S(\beta_S, w) = S$, and (we use the form of $\mathfrak{T}^+ = \mathfrak{T} \otimes {}^{<\omega}\omega$ here) $\beta_S \cap T'_n = \beta_S \cap \beta_{S'} = \{t \in T^+ : t \leq w\}$ for distinct $S, S' \in \mathcal{S}_w$. Let $B_w = \{\beta_S : S \in \mathcal{S}_w\}$, and $T_w = \bigcup B_w$. We can assume that $\sigma \in B_{w_0}$. Note that $|B_w| \leq \xi(\varphi)$. Now set $T'_{n+1} = T'_n \cup \bigcup \{T_w : w \in T'_n,\ ht(w) = n\}$. Finally define $T' = \bigcup_{n<\omega} T'_n$. Note that $\sigma \subseteq T'$ and $v \in T'$.

Let $\mathfrak{M}' = \langle \mathfrak{T}', D, I' \rangle$ and $\mathfrak{T}'$ be the restrictions of $\mathfrak{M}$ and $\mathfrak{T}^+$ to $T'$. One can easily show by induction on the construction of $\psi \in sub(\varphi)$ that $(\mathfrak{M}, \beta, w) \models^{\mathfrak{a}} \psi$ iff $(\mathfrak{M}', \beta, w) \models^{\mathfrak{a}} \psi$, for all full branches $\beta$ in $\mathfrak{T}'$ and all $w \in \beta$. (For example, suppose $(\mathfrak{M}, \beta, w) \models^{\mathfrak{a}} \mathsf{E}\psi$. Then there is $\beta' \in B(w)$ in $\mathfrak{T}^+$ such that

$(\mathfrak{M},\beta',w) \models^{\mathfrak{a}} \psi$. Pick a full branch $\gamma$ in $\mathfrak{T}'$ for which $S(\beta',w) = S(\gamma,w)$. Since $\psi$ is a sentence, we have $(\mathfrak{M},\gamma,w) \models^{\mathfrak{a}} \psi$. It follows by IH that $(\mathfrak{M}',\gamma,w) \models^{\mathfrak{a}} \psi$ and so $(\mathfrak{M}',\beta,w) \models^{\mathfrak{a}} \mathsf{E}\psi$.)

Thus $\mathfrak{M}'$ satisfies $\varphi$ and we can work with this model instead of $\mathfrak{M}$. Define an equivalence relation $\sim_w$ on $B(w)$ (defined in $\mathfrak{T}'$ now), for $w \in T'$, by taking $\beta \sim_w \beta'$ when $(\mathfrak{M}',\beta,w) \models^{\mathfrak{a}} \psi$ iff $(\mathfrak{M}',\beta',w) \models^{\mathfrak{a}} \psi$, for every $\psi \in sub(\varphi)$ and every assignment $\mathfrak{a}$. The $\sim_w$-equivalence class generated by $\beta$ will be denoted by $[\beta]_w$.

Since only $\bigcirc$ is applied to open formulas, we can show that the number of $\sim_w$-equivalence classes is bounded by $\rho(\varphi)$. To show this, for $w \in T'$, full branches $\beta$ and $\beta'$ in $B(w)$, and $d < \omega$, we put $\beta \sim_w^d \beta'$ if for all $t \in T'$ with $t \geq w$ and $ht(t) \leq ht(w) + d$, we have

1. $t \in \beta$ iff $t \in \beta'$,
2. if $t \in \beta$, then for all assignments $\mathfrak{a}$ and all $\psi \in sub(\varphi)$ with at most $ht(w)+d-ht(t)$ occurrences of $\bigcirc$, we have $(\beta,t) \models^{\mathfrak{a}} \psi$ iff $(\beta',t) \models^{\mathfrak{a}} \psi$.

An induction on $d$ shows that the number $\sharp(d)$ of $\sim_w^d$-classes is at most $\xi(\varphi)^d \cdot 2^{(d+1)|sub(\varphi)|}$ (for any $w$). For $d = 0$, one may check that if $(\beta,w) \models \psi$ iff $(\beta',w) \models \psi$ for each sentence $\psi \in sub(\varphi)$, then $\beta \sim_w^0 \beta'$. So $\sharp(0) \leq 2^{|sub(\varphi)|}$. Assume the result for $d$. One may check that if $\beta,\beta' \in B(w)$ contain a common immediate successor $v$ of $w$, $(\beta,w) \models \psi$ iff $(\beta',w) \models \psi$ for each sentence $\psi \in sub(\varphi)$, and $\beta \sim_v^d \beta'$, then $\beta \sim_w^{d+1} \beta'$. Both checks involve an induction on $\psi$ in (2) above. It follows that $\sharp(d+1) \leq \xi(\varphi) \cdot 2^{|sub(\varphi)|} \cdot \sharp(d)$, and hence that $\sharp(d) \leq \xi(\varphi)^d \cdot 2^{(d+1)|sub(\varphi)|}$ for all $d$, as required. Finally observe that if $\beta \sim_w^{|sub(\varphi)|} \beta'$ then $\beta \sim_w \beta'$, so that $\sim_w$ has at most $\sharp(|sub(\varphi)|) \leq \rho(\varphi)$ classes.

Let $\beta_1^w, \dots \beta_{n_w}^w$ be some minimal list of full branches such that $\{[\beta_1^w]_w, \dots, [\beta_{n_w}^w]_w\}$ is the set of all $\sim_w$-equivalence classes. With each $a \in D$ we associate a trace

$$\tau_a^w : \{1,\dots,n_w\} \to \bigcup \mathcal{S}_w$$

by taking $\tau_a^w(i) = \mathrm{tp}(\beta_i^w,w,a)$. Denote the resulting set of traces by $\mathcal{T}_w$. Let $\Theta_w = (\mathcal{S}_w,\mathcal{T}_w)$ for all $w \in T'$. We are now in a position to define a quasimodel $f$ over $\mathfrak{T}'$ satisfying $\varphi$. If $w$ is not the root, then set $f(w) = ((\Theta_v,\Theta_w,R_w,N_w),g_w)$, where $v$ is the immediate predecessor of $w$, and for root $w_0$ let $f(w_0) = (\Theta_{w_0},g_{w_0})$, where

- $g_w(\beta) = i$ iff $\beta \in [\beta_i^w]_w$,
- $R_w = \{(\tau_a^v,\tau_a^w) : a \in D\}$,
- $N_w = \{(g_v(\beta),g_w(\beta)) : \beta \in B(w)\}$.

(Let us show that $f$ is a quasimodel. It should be clear that the first component of each $f(w)$ ($w_0 \neq w$) is a connection and that of $f(w_0)$ is a realisable state candidate. We will check only item 2 of Definition 14. Suppose $g_w(\beta) = i$ and $\chi \mathsf{U} \psi \in \Theta_w(i)$. Since $\chi \mathsf{U} \psi$ is a sentence, this means that $\chi \mathsf{U} \psi \in \tau_a^w(i) = \mathrm{tp}(\beta_i^w,w,a) = \mathrm{tp}(\beta,w,a)$ for every $a \in D$. So there is $u > w$ with $u \in \beta$ and $\psi \in \mathrm{tp}(\beta,u,a)$ (from which $\psi \in \Theta_u(g_u(\beta))$ follows) and for all $v \in (w,u)$ we have $\chi \in \mathrm{tp}(\beta,v,a)$ (from which $\chi \in \Theta_v(g_v(\beta))$ follows). The converse implication is proved similarly.)

($\Leftarrow$) Now suppose that $f$ is a quasimodel for $\varphi$ over $\mathfrak{T} = \langle T,<\rangle$ with root $w_0$. Let $f(w_0) = (\Theta_{w_0},g_{w_0})$ and $f(w) = (C_w,g_w) = ((\Delta_w,\Theta_w,R_w,N_w),g_w)$ for non-root $w \in T$. Let $\Theta_w = (\mathcal{S}_w,\mathcal{T}_w)$ and $n_w = n_{\Theta_w}$.

A *run* $r$ in $f$ is a function associating with any $w \in T$ a trace $r(w) \in \mathcal{T}_w$ such that $(r(v),r(w)) \in R_w$ for any non-root $w$ with immediate predecessor $v$. Using the condition that the range and domain of

$R_w$ coincide with $\{1,\dots,n_w\}$ and $\{1,\dots,n_v\}$, respectively, it is not difficult to see that, for any $w$ and any $\tau \in \mathcal{T}_w$, there exists a run $r$ with $r(w) = \tau$. Let $\mathcal{R}$ be the set of all runs.

For every $w \in T$ we find a first-order structure $I(w)$ with domain $D = \mathcal{R}$ realising $\Theta_w = (\mathcal{S}_w, \mathcal{T}_w)$ and such that for all $i \in \{1,\dots,n_w\}$, $r \in D$, and $\psi \in sub(\varphi)$,

$$\psi \in r(w)(i) \quad \text{iff} \quad I(w) \models \psi^i[r].$$

Let $\mathfrak{M} = \langle \mathfrak{T}, D, I\rangle$ and let $\mathfrak{a}$ be any assignment in $D$. We show by induction that for all $\psi \in sub(\varphi)$, all $w \in T$, and all $\beta \in B(w)$ with $g_w(\beta) = i$, say, we have

$$I(w) \models^{\mathfrak{a}} \psi^i \quad \text{iff} \quad (\mathfrak{M}, \beta, w) \models^{\mathfrak{a}} \psi.$$

Suppose first that $\psi = \chi_1 \mathsf{U} \chi_2$. Then $\psi$ is a sentence, and for all $r \in \mathcal{R}$ we have

$$\begin{aligned}
I(w) \models^{\mathfrak{a}} p^i_\psi \;&\Leftrightarrow\; \chi_1 \mathsf{U} \chi_2 \in r(w)(g_w(\beta))\\
&\Leftrightarrow\; \exists u > w\, \big(u \in \beta \wedge \chi_2 \in r(u)(g_u(\beta)) \wedge\\
&\qquad \forall v \in (w,u)\,(\chi_1 \in r_v(g_v(\beta)))\big) \text{ (since $r$ is a run and $\psi$ is a sentence)}\\
&\Leftrightarrow\; \exists u > w\,(u \in \beta \wedge I(u) \models^{\mathfrak{a}} \chi_2^{g_u(\beta)} \wedge \forall v \in (w,u)\, I(v) \models^{\mathfrak{a}} \chi_1^{g_v(\beta)})\\
&\Leftrightarrow\; \exists u > w(u \in \beta \wedge (\mathfrak{M}, \beta, u) \models^{\mathfrak{a}} \chi_2 \wedge \forall v \in (w,u)(\mathfrak{M}, \beta, v) \models^{\mathfrak{a}} \chi_1) \text{ (by IH)}\\
&\Leftrightarrow\; (\mathfrak{M}, \beta, w) \models^{\mathfrak{a}} \psi.
\end{aligned}$$

The case of $\psi = \chi_1 \mathsf{S} \chi_2$ is a mirror image. Now suppose $\psi = \bigcirc\chi$. Then for any $r \in \mathcal{R}$,

$$\begin{aligned}
I(w) \models P^i_\psi[r] \;&\Leftrightarrow\; \bigcirc\chi \in r(w)(g_w(\beta))\\
&\Leftrightarrow\; \chi \in r(v)(g_v(\beta)), \text{ for the immediate successor $v$ of $w$ in $\beta$,}\\
&\qquad \text{since $C_v$ is a connection and $r$ a run}\\
&\Leftrightarrow\; I(v) \models \chi^{g_v(\beta)}[r]\\
&\Leftrightarrow\; (\mathfrak{M}, \beta, v) \models \chi[r] \text{ (by IH)}\\
&\Leftrightarrow\; (\mathfrak{M}, \beta, w) \models \bigcirc\chi[r].
\end{aligned}$$

For $\psi = \mathsf{E}\chi$:

$$\begin{aligned}
I(w) \models^{\mathfrak{a}} p^i_\psi \;&\Leftrightarrow\; \mathsf{E}\chi \in r(w)(g_w(\beta))\\
&\Leftrightarrow\; \chi \in r(w)(g_w(\beta')), \text{ for some $\beta' \in B(w)$, since $\Theta_w$ is}\\
&\qquad \text{a state candidate, $g_w$ is surjective and $\psi$ is a sentence}\\
&\Leftrightarrow\; \exists \beta'\,(\beta' \in B(w) \wedge I(w) \models^{\mathfrak{a}} \chi^{g_w(\beta')})\\
&\Leftrightarrow\; \exists \beta'\,(\beta' \in B(w) \wedge (\mathfrak{M}, \beta', w) \models^{\mathfrak{a}} \chi) \text{ (by IH)}\\
&\Leftrightarrow\; (\mathfrak{M}, \beta, w) \models \mathsf{E}\chi.
\end{aligned}$$

Since $\varphi \in r(w)(g_w(\sigma))$ for some $w \in T$, $\sigma \in B(w)$ and $r \in \mathcal{R}$, we finally obtain $(\mathfrak{M}, \sigma, w) \models \varphi$. □

Now we construct a reduction of $\mathcal{QPCTL}^*_w$ to non-local $\mathcal{PCTL}^*$ by means of encoding quasi-models in non-local propositional tree models. Suppose again that a $\mathcal{QPCTL}^*_w$-sentence $\varphi$ is fixed.

With every realisable state candidate $\Theta = (\mathcal{S}, \mathcal{T})$ for $\varphi$, every connection $C$, every $i \le \rho(\varphi)$, and every sentence $\chi \in sub(\varphi)$, which either is a propositional variable or starts with an existential quantifier $\exists x$, we associate propositional variables $p_\Theta$, $p_C$, $p_i$, and $p_\chi$, respectively. Let $\cdot^\sharp$ be a translation from the set of $\mathcal{QPCTL}^*_w$-sentences into the set of $\mathcal{PCTL}^*$-formulas which distributes over the booleans, temporal operators and path quantifiers, and $\chi^\sharp = p_\chi$, where $\chi$ is a propositional variable or a sentence of the form $\exists x \vartheta$. Clearly, $\varphi^\sharp$ is a $\mathcal{PCTL}^*$-formula. Let

$$\varphi^\star = \varphi^\sharp \wedge \big((\chi \wedge \neg\Diamond_P \top) \vee \Diamond_P(\chi \wedge \neg\Diamond_P \top)\big),$$

where $\chi$ is the conjunction of the formulas (1)–(7) defined below.

$$\bigvee_{\Theta\in\mathcal{R}(\varphi)} \mathsf{A}p_\Theta \wedge \bigwedge_{\Theta\neq\Theta'} \mathsf{A}(p_\Theta \to \neg p_{\Theta'}) \qquad \mathsf{A}\Box_F\Big(\bigvee_{C\in\mathcal{C}(\varphi)} \mathsf{A}p_C \wedge \bigwedge_{C\neq C'} \mathsf{A}(p_C \to \neg p_{C'})\Big) \tag{1}$$

Here $\mathcal{R}(\varphi)$ and $\mathcal{C}(\varphi)$ are the sets of realisable state candidates and connections for $\varphi$, respectively. The formulas in (1) say that the $p_\Theta$ and $p_C$ are 'local' (so we can write $w \models p_\Theta$ and $w \models p_C$) and that precisely one $p_\Theta$ holds at the root and precisely one $p_C$ holds at each non-root point.

Intuitively, $w \models p_C$ means that $f(w) = (C, g)$, for some $g$. Say that a pair of connections $(C_1, C_2)$ is *suitable* if the second state candidate of $C_1$ coincides with the first state candidate of $C_2$. The set of all suitable pairs of connections is denoted by $\mathcal{C}_s(\varphi)$. A pair $(\Theta, C)$ is *suitable* if the first state candidate of $C$ coincides with $\Theta$. The set of all suitable pairs of this form is denoted by $\mathcal{R}_s(\varphi)$. The following formulas say that the pair induced by a point and its immediate predecessor is suitable:

$$\mathsf{A} \bigvee_{(\Theta,C)\in\mathcal{R}_s(\varphi)} (p_\Theta \wedge \bigcirc p_C), \qquad \mathsf{A}\Box_F \bigvee_{(C_1,C_2)\in\mathcal{C}_s(\varphi)} (p_{C_1} \wedge \bigcirc p_{C_2}) \tag{2}$$

Intuitively, for $i$ such that $1 \leq i \leq \rho(\varphi)$, $(\beta, w) \models p_i$ is intended to mean $g_w(\beta) = i$. This is ensured by the formulas

$$\mathsf{A} \bigwedge_{1\leq i<j\leq\rho(\varphi)} (p_i \to \neg p_j), \quad \mathsf{A}\Box_F \bigwedge_{1\leq i<j\leq\rho(\varphi)} (p_i \to \neg p_j) \tag{3}$$

$$\bigwedge_{\Theta\in\mathcal{R}(\varphi)} \Big(p_\Theta \to \bigwedge_{1\leq i\leq n_\Theta} \mathsf{E}p_i \wedge \mathsf{A} \bigvee_{1\leq i\leq n_\Theta} p_i\Big), \qquad \mathsf{A}\Box_F \bigwedge_{C\in\mathcal{C}(\varphi)} \Big(p_C \to \bigwedge_{1\leq i\leq n_\Theta} \mathsf{E}p_i \wedge \mathsf{A} \bigvee_{1\leq i\leq n_\Theta} p_i\Big) \tag{4}$$

Here and below we assume that $C = (\Delta, \Theta, R, N)$. Now we write down a formula which says that $N$ in $C$ is determined by the functions $g_w$:

$$\mathsf{A}\Box_F \bigwedge_{C\in\mathcal{C}(\varphi)} \Big(p_C \to \big(\bigwedge_{(i,j)\in N} \mathsf{E}(p_j \wedge \bigcirc_P p_i) \wedge \mathsf{A} \bigvee_{(i,j)\in N} (p_j \wedge \bigcirc_P p_i)\big)\Big) \tag{5}$$

Finally, we have to ensure that the set of sentences true at $(\beta, w)$ corresponds to the set of sentences in $\Theta_w(g_w(\beta))$. Let $\Theta'_w(i)$ be the set of sentences in $sub(\varphi) \setminus \Theta_w(i)$. Then we include:

$$\mathsf{A} \bigwedge_{\Theta\in\mathcal{R}(\varphi), 1\leq i\leq n_\Theta} \Big((p_\Theta \wedge p_i) \to \big(\bigwedge_{\chi\in\Theta(i)} \chi^\sharp \wedge \bigwedge_{\chi\in\Theta'(i)} \neg\chi^\sharp\big)\Big) \tag{6}$$

$$\mathsf{A}\Box_F \bigwedge_{C\in\mathcal{S}(\varphi), 1\leq i\leq n_\Theta} \Big((p_C \wedge p_i) \to \big(\bigwedge_{\chi\in\Theta(i)} \chi^\sharp \wedge \bigwedge_{\chi\in\Theta'(i)} \neg\chi^\sharp\big)\Big) \tag{7}$$

THEOREM 16. *A $\mathcal{QPCTL}^*_w$-sentence $\varphi$ is satisfiable in a full model iff the $\mathcal{PCTL}^*$-formula $\varphi^\star$ is satisfiable in a full non-local model.*

*Proof.* ($\Rightarrow$) If $\varphi$ is satisfiable, then it is satisfied in a quasimodel $f$ for $\varphi$ based on an $\omega$-tree $\mathfrak{T} = \langle T, < \rangle$. Let $f(w) = (C_w, g_w) = ((\Delta_w, \Theta_w, N_w, R_w), g_w)$ if $w$ in not the root and $f(w_0) = (\Delta_{w_0}, g_{w_0})$ for root $w_0$ of $\mathfrak{T}$. Define a (propositional) valuation $h$ in $\mathfrak{T}$ by taking, for all $w \in T$ and $\beta \in B(w)$:

- $(\beta, w) \in h(p_\Theta)$ iff $\Theta = \Theta_w$, for every realisable state candidate $\Theta$;
- $(\beta, w) \in h(p_C)$ iff $C = C_w$, for every connection $C$;
- $(\beta, w) \in h(p_i)$ iff $g_w(\beta) = i$, for all $i < \rho(\varphi)$;
- $(\beta, w) \in h(p_\chi)$ iff $\chi \in \Theta_w(g_w(\beta))$, for all sentences $\chi \in sub(\varphi)$.

Let us prove that the full model $\mathfrak{M} = \langle \mathfrak{T}, h \rangle$ is as required. First we show by induction that for all sentences $\chi \in sub(\varphi)$ and all $\beta \in B(w)$,

$$\chi \in \Theta_w(g_w(\beta)) \quad \text{iff} \quad (\beta, w) \models \chi^\sharp. \tag{8}$$

The basis of induction (propositional atoms and sentences $\exists x \chi$) and the boolean case are trivial. So suppose $\chi$ is of the form $\chi \mathsf{U} \psi$ or $\mathsf{E}\psi$ (recall that every sentence of the form $\bigcirc\psi$ is replaced by $\bot \mathsf{U} \psi$).

Let $\chi \mathsf{U} \psi \in \Theta_w(g_w(\beta))$. Then there is $u > w$ with $u \in \beta$ such that $\psi \in \Theta_u(g_u(\beta))$ and for all $v \in (w, u)$ we have $\chi \in \Theta_v(g_v(\beta))$. It follows by IH that $(\beta, u) \models \psi^\sharp$ and $(\beta, v) \models \chi^\sharp$, for all $v \in (w, u)$. Therefore $(\beta, w) \models (\chi \mathsf{U} \psi)^\sharp$, since $(\chi \mathsf{U} \psi)^\sharp = \chi^\sharp \mathsf{U} \psi^\sharp$. The converse implication is proved analogously.

Suppose now that $\mathsf{E}\psi \in \Theta_w(g_w(\beta))$. Then there is $j$ such that $\psi \in \Theta_w(j)$. There is $\beta' \in B(w)$ for which $g_w(\beta') = j$. Therefore $\psi \in \Theta_w(g_w(\beta'))$. It follows by IH that $(\beta', w) \models \psi^\sharp$ and consequently $(\beta, w) \models \mathsf{E}\psi^\sharp$. So we have $(\beta, w) \models (\mathsf{E}\psi)^\sharp$. The other direction is similar.

Since $\varphi \in \Theta_w(g_w(\beta))$, for some $w \in T$ and $\beta \in B(w)$, we have that $(\mathfrak{M}, \beta, w) \models \varphi^\sharp$.

Now we show that $(\beta, w) \models (\chi \wedge \neg \Diamond_P \top) \vee \Diamond_P(\chi \wedge \neg \Diamond_P \top)$, i.e., $(\beta, w_0) \models \chi$, where $w_0$ is the root of $\mathfrak{T}$. We only check formulas (5), (6) and (7).

(5) Suppose $(\gamma, v) \models p_C$, for a full branch $\gamma$ of $\mathfrak{T}$ and some $v \in \gamma$ such that $v \neq w_0$. Then by (1) $C = C_v$. According to Definition 14, for each pair $(i, j) \in N_v$ there is a branch in $B(v)$, say $\gamma'$, such that $g_v(\gamma') = j$ and $g_u(\gamma') = i$, for the immediate predecessor $u$ of $v$. Hence, $(\gamma', v) \models p_j$ and $(\gamma', u) \models p_i$, from which $(\gamma, v) \models \mathsf{E}(p_j \wedge \bigcirc_P p_i)$ for all $(i, j) \in N_v$. Definition 14 also says that for each branch $\gamma' \in B(v)$, there is a pair $(i, j) \in N_v$ such that $g_v(\gamma') = j$ and $g_u(\gamma') = i$, where $u$ is the immediate predecessor of $v$. This means that $(\gamma', v) \models p_j$ and $(\gamma', u) \models p_i$. So $(\gamma, v) \models \mathsf{A} \bigvee_{(i,j) \in N_v} (p_j \wedge \bigcirc_P p_i)$.

(6) By (8), if $(\beta, w_0) \models p_\Theta$ and $(\beta, w_0) \models p_i$, for a state candidate $\Theta$ and some $i \leq n_\Theta$, then $(\beta, w_0) \models \chi^\sharp$ iff $\chi \in \Theta_{w_0}(g_{w_0}(\beta))$. (7) is considered similarly using (1) and (2).

($\Leftarrow$) Conversely, suppose $\mathfrak{M} = \langle \mathfrak{T}, h \rangle$ satisfies $\varphi^\star$. Then $\chi$ is true at the root $w_0$ of $\mathfrak{T}$. Define a quasimodel $f$ by taking $f(w) = (C_w, g_w) = ((\Delta_w, \Theta_w, N_w, R_w), g_w)$ if $w \neq w_0$, and $f(w_0) = (\Theta_{w_0}, g_{w_0})$, where

- $\Theta_{w_0}$ is the unique $\Theta$ for which $w_0 \models p_\Theta$ (this is independent of the branch of evaluation);
- for $w \neq w_0$, $C_w$ is the unique $C$ such that $w \models p_C$;
- $g_w(\beta) = i$ for the unique $i$ for which $(\beta, w) \models p_i$.

We show that $f$ is a quasimodel by checking the conditions of Definition 14. The first condition follows from the definition of suitable pair, (1) and (2). Suppose now that $(i, j) \in N_w$. We have $w \models p_{C_w}$. Hence, by (5), $w \models \mathsf{E}(p_j \wedge \bigcirc_P p_i)$. So there is $\sigma \in B(w)$ with $(\sigma, w) \models p_j \wedge \bigcirc_P p_i$. This implies $g_w(\sigma) = j$ and $g_v(\sigma) = i$, for the immediate predecessor $v$ of $w$. Conversely, suppose $g_w(\sigma) = j$

and $g_v(\sigma) = i$, where $v$ is the immediate predecessor of $w$. Then $(\sigma, v) \models p_i$ and $(\sigma, w) \models p_j$. Hence $w \models \mathsf{E}(p_j \wedge \bigcirc_P p_i)$ and so, by (5), $(i, j) \in N_w$. Thus, the second condition of Definition 14 holds. To prove the third one, we first show that for every sentence $\chi \in sub(\varphi)$, every $w \in T$ and every $\sigma \in B(w)$,

$$\chi \in \Theta_w(g_w(\sigma)) \quad \text{iff} \quad (\sigma, w) \models \chi^\sharp.$$

Suppose $g_w(\sigma) = i$. Then $(\sigma, w) \models p_i$. If $w = w_0$ then $(\sigma, w) \models p_{\Theta_w} \wedge p_i$ and, by (6),

$$(\sigma, w) \models \bigwedge_{\chi \in \Theta_w(i)} \chi^\sharp \wedge \bigwedge_{\chi \notin \Theta_w(i)} \neg \chi^\sharp.$$

The case $w \neq w_0$ is proved using (7). Now we have:

$$\begin{aligned}
\chi_1 \mathsf{U} \chi_2 \in \Theta_w(g_w(\sigma)) \quad &\Leftrightarrow \quad (\sigma, w) \models (\chi_1 \mathsf{U} \chi_2)^\sharp \\
&\Leftrightarrow \quad (\sigma, w) \models \chi_1^\sharp \mathsf{U} \chi_2^\sharp \\
&\Leftrightarrow \quad \exists u > w \big( u \in \sigma \wedge (\sigma, u) \models \chi_2^\sharp \wedge \forall v \in (w, u)\, (\sigma, v) \models \chi_1^\sharp \big) \\
&\Leftrightarrow \quad \exists u > w \big( u \in \sigma \wedge \chi_2 \in \Theta_u(g_u(\sigma)) \wedge \forall v \in (w, u)\, \chi_1 \in \Theta_v(g_v(\sigma)) \big).
\end{aligned}$$

The last condition is the mirror image of the previous one. □

# 4 Conclusion

The decidability result for the weak one-variable fragment of first-order $\mathsf{CTL}^*$ can be used to obtain decidability results for certain spatio-temporal logics connecting $\mathsf{CTL}^*$ with region connection calculus RCC8 (see the survey paper [5]). From this viewpoint it has sufficient expressive power to be useful. However, there is still a gap between the undecidability of the one-variable fragment of first-order $\mathsf{CTL}^*$ and the decidability of its weak one-variable fragment. In particular, the following problems are still open. What happens if the path-quantifier $\mathsf{E}$ is applied to open formulas as well? Or, what happens if all temporal operators are applied to open formulas (but $\mathsf{E}$ only to sentences)?

Another open problem is the computational complexity of the logics considered here. From the reduction proofs provided in the present paper we obtain only non-elementary decision procedures. We do not believe that this is optimal.

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