Multi-Dimensional Modal Logic as a Framework for Spatio-Temporal Reasoning

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Abstract. In this paper we advocate the use of multi-dimensional modal logics as a framework for knowledge representation and, in particular, for representing spatio-temporal information. We construct a two-dimensional logic capable of describing topological relationships that change over time. This logic, called **PSTL** (Propositional Spatio-Temporal Logic) is the Cartesian product of the well-known temporal logic **PTL** and the modal logic $\mathbf{S4_u}$, which is the Lewis system **S4** augmented with the universal modality. Although it is an open problem whether the full **PSTL** is decidable, we show that it contains decidable fragments into which various temporal extensions (both point-based and interval based) of the spatial logic **RCC-8** can be embedded. We consider known decidability and complexity results that are relevant to computation with muli-dimensional formalisms and discuss possible directions for further research.

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1. Introduction

It is widely accepted that many kinds of AI applications require high-level reasoning involving spatial and temporal concepts (see e.g. (Hayes, 1979; Hobbs and Moore, 1985; Davis, 1990)). Several approaches have been applied to representing these concepts: some researchers have developed specialised computation-oriented representations such as solid geometry (Requicha, 1980), constraints techniques (Allen, 1981) and spatio-temporal database architectures (Güting et al., 2000); others have employed techniques of formal logics. Typically, logical representations have made use of the expressive power of first-order languages (e.g. (McCarthy and Hayes, 1969; Clarke, 1981; Allen, 1984; Hayes, 1985; Kowalski and Sergot, 1986; Randell et al., 1992b)); but, since first-order logic is undecidable, such languages do not provide an effective reasoning algorithm unless supplemented by some special purpose inference mechanism.

An alternative to first-order logic is the framework of *propositional modal logic*, which extends classical propositional calculus with one or more 'modal' operators, interpreted on some relational or algebraic structures. Appropriate structures may be chosen to correspond to an aspect of reality, such as time or space, which one wishes to describe within a logical representation.

For example, one may think of time as a sequence of points represented by the natural numbers $\langle \mathbb{N}, < \rangle$ on which temporal operators — 'some time in the future', 'since' etc. — can be defined. For instance $\Diamond^+\varphi$ is true at a time point $t \in \mathbb{N}$ (in symbols $t \models \Diamond^+\varphi$) if and only if $t' \models \varphi$ for some $t' > t$. The resulting *propositional temporal* *logic*, **PTL**, is decidable in PSPACE (Sistla and Clarke, 1985) and by Kamp's theorem (Kamp, 1968) it is as expressive (on $\langle \mathbb{N}, < \rangle$) as the first-order language with unary predicates $P_i(x)$, $<$ and $=$. Temporal logics of this sort are widely used in computer science (Manna and Pnueli, 1992; Stirling, 1992).

Modal representations of space are relatively new to computer science. However, the possibility of giving topological interpretations of intuitionistic propositional logic has been known since the thirties (Stone, 1937–8; Tarski, 1938). Tang (1938) and later McKinsey and Tarski (1948) interpreted the necessity operator of the Lewis modal system **S4** as the interior operator of topological spaces and proved that **S4** is sound and complete with respect to this interpretation (for details see e.g. Chagrov and Zakharyaschev (1997)). Since **S4** is decidable in PSPACE, this suggests that it might provide a computationally viable language for describing topological information. That **S4** really can be used for qualitative spatial representation and reasoning was observed by Bennett (1996, 1998), who embedded the spatial language **RCC-8** into $\mathbf{S4_u}$ — i.e. **S4** extended with the universal modality $\forall$ (for all points in the space). This encoding, which will be explained in Section 2, provides a decision procedure for a spatial language that can express a large class of significant topological relations including all those in **RCC-8**.

As we have seen, propositional modal and temporal logics may be decidable and hence amenable to automated reasoning (Demri, 1994; Giunchiglia and Sebastiani, 1996; Horrocks, 1998; Dixon et al., 1998; Voronkov, 1999; De Nivelle et al., 2000). This of course means they are less expressive than first-order logic. Moreover it is clear that modalities of single type (say, only temporal or only spatial) are not comprehensive enough for many real applications, which require reasoning with a range of different concepts. What we actually need is systems combining several types of modal operator. Such *multi-modal* logics have been the subject of much recent research — e.g. (Kracht and Wolter, 1991; Spaan, 1993; Marx and Venema, 1996; Gabbay, 1998; Wolter, 2000; Gabbay et al., 2000). Central problems in this field concern how modalities can be combined, whether interplay between operators preserves decidabilty and other meta-logical properties, and what is the computational complexity of the resulting systems.

Whereas modalities can be successfully combined as logically independent operators (Kracht and Wolter, 1991; Fine and Schurz, 1996), a key feature of most of the more interesting systems is the *interaction* between different modalities. This applies in particular to modal operators relevant to describing spatial and temporal concepts. Consider, for instance, a moving region (the black disk) depicted in Fig. 1. If we can model space as an $\mathbf{S4_u}$-frame (i.e., a quasi-order) $\mathfrak{F}$ and if the flow of time is represented by a linear order $\mathfrak{G}$, then the whole spatio-temporal 'universe' can be viewed as the Cartesian product $\mathfrak{F} \times \mathfrak{G}$, in which the $\mathbf{S4_u}$-modalities act 'horizontally' to talk about spatial regions, while the temporal operators act 'vertically' taking care of their movements in time.

Cartesian products of Kripke frames are typical examples of multi-dimensional structures that serve as models of *multi-dimensional modal logics*. The idea of a multi-dimensional modal logic was introduced by Segerberg (1973) who presented an axiomatisation for a 2-dimensional extension of **S5**. Products of modal logics were investigated in (Sheht-

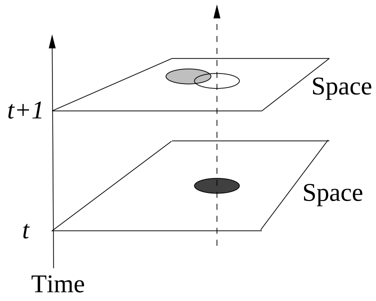

Figure 1.

man, 1978; Gabbay and Shehtman, 1998; Marx, 1999). In computer science and AI, multi-dimensional modal logics are used e.g. for constructing temporal epistemic logics for multi-agent and distributed systems (Fagin et al., 1995), temporal logics of parallel processes (Reynolds, 1997), temporal, epistemic, and dynamic description logics (Baader and Ohlbach, 1995; Wolter and Zakharyaschev, 1998, 2000c, 1999). Further details and development of multi-dimensional modal logic can be found in (Gabbay et al., 2000).

Although the idea of constructing multi-dimensional modal logics may appear natural and simple, the resulting 'hybrids' often turn out to be very complex or undecidable, even if the one-dimensional components are 'almost tractable'. As with other frameworks for knowledge representation there is a delicate balance between expressive power and computational complexity. Let us consider, for instance, the following naïve example of a 'compass' spatial logic (Venema, 1990).

In our everyday practice, we often connect spatial structures with this or that system of coordinates. For instance, in geographical maps we have four compass directions: North, South, West, and East. Using these we can say e.g. that Moscow is to the North-East of London.

Spatial relations of this sort can be expressed in a modal language with four operators $\Diamond_N$ ('somewhere to the North'), ..., $\Diamond_S$ ('somewhere to the South') interpreted in the Cartesian product of two linear orders, say, $\langle \mathbb{R}, < \rangle \times \langle \mathbb{R}, < \rangle$ in the standard Kripke-style manner:

$$(x, y) \models \Diamond_N \varphi \quad \text{iff} \quad \exists y' > y \; (x, y') \models \varphi,$$

$$(x, y) \models \Diamond_W \varphi \quad \text{iff} \quad \exists x' < x \; (x', y) \models \varphi,$$

etc.

That 'Moscow is to the North-East of London' can be represented in this language by the formula $\mathsf{London} \rightarrow \Diamond_N \Diamond_E \mathsf{Moscow}$. It is well known from modal logic that the satisfiability problem for a language with a single $\Diamond$ operator is co-NP-complete for the frames $\langle \mathbb{R}, < \rangle$ or $\langle \mathbb{N}, < \rangle$, (Ono and Nakamura, 1980). However, the satisfiability problem for a bimodal language (with $\Diamond_N$ and $\Diamond_E$) in $\langle \mathbb{R}, < \rangle \times \langle \mathbb{R}, < \rangle$ or $\langle \mathbb{N}, < \rangle \times \langle \mathbb{N}, < \rangle$ turns out to be *undecidable*, even Σ_1^1-complete (Spaan, 1993; Marx and Reynolds, 1999; Reynolds and Zakharyaschev, 2000). The interval temporal logic of Halpern and Shoham (1986) can be interpreted in the $\{(x, y) \in \mathbb{N} \times \mathbb{N} : x \leq y\}$ part of $\langle \mathbb{N}, < \rangle \times \langle \mathbb{N}, < \rangle$ (see (Venema, 1990)), and this logic is also not recursively enumerable. There are many other examples of undecidable multi-dimensional modal logics, e.g. $\mathbf{S5}^3$ (Maddux, 1980) or even any logic between $\mathbf{K}^3$ and $\mathbf{S5}^3$ (Hirsch et al., 2000).

Thus, we see that multi-dimensional modal logics are not easy to deal with, and we need to be careful in constructing effective and expressive spatio-temporal formalisms. For example, the straightforward attack on the problem by means of using the Cartesian products of frames for **S4** and the flow of time $\langle \mathbb{N}, < \rangle$ (or any other infinite linear order) has not brought any result yet: whether the logic of such 2-dimensional frames is decidable remains one of the challenging open problems in the field.

In the rest of this paper, we discuss possible uses of multi-dimensional modal logics for spatio-temporal representation and reasoning, giving initial results already obtained in this direction. Thereby we would like both to attract the attention of the knowledge representation community to this novel and promising approach and also to give incentive to logicians to investigate modal languages that may turn out to have practical utility.

2. Region Connection Calculus and its Modal Interpretation

The *Region Connection Calculus* (**RCC**) is a first-order theory proposed by Randell et al. (1992a) for qualitative spatial representation and reasoning.[1] The basic language of **RCC** contains only one primitive predicate $\mathsf{C}(X, Y)$, read as 'region X *is connected* with region Y'.[2]

Many other spatial relations can be defined in terms of the C primitive. Of particular significance are the eight relations depicted in Fig. 2. These form a pairwise disjoint and exhaustive set of relations known as **RCC-8**. Essentially the same set (but applied to the more restricted class of connected regions) has been independently identified as useful in the context of Geographical Information Systems (Egenhofer and

[1] A comprehensive account of this theory and its applications can be found in (Cohn et al., 1997).

[2] Randell et al. (1992a) also considered an extended language incorporating an additional primitive function $\mathsf{conv}(r)$, which denotes the *convex-hull* of region r. We will comment briefly on convexity later (Section 6).

Franzosa, 1991). In English, the relations can be described as: DisConnection, External Connection, Partial Overlap, Tangential Proper Part, Non-Tangential Proper Part and Equality. Formal notations for these relations are given under the diagrams. The part relations are asymmetric, so each has an inverse, suffixed by 'i'.

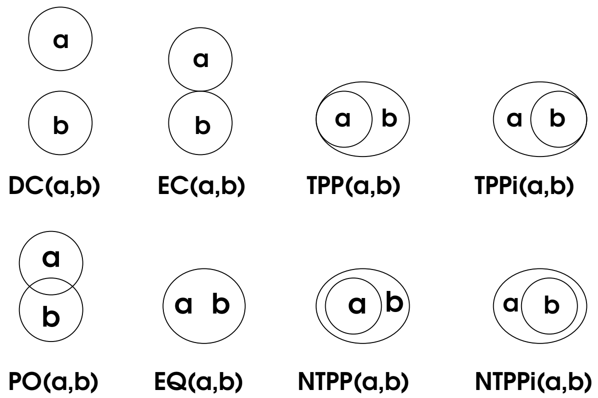

Figure 2. Basic Relations in the RCC Theory

The **RCC** formalism was originally presented as a naïve theory in the spirit of Hayes (1979), so no specific model was assumed. However, it has been found that the theory can be interpreted in classical point-set topology (Gotts 1996a; Bennett 1997,1998). Thus, we can take as models of **RCC**, topological spaces, $\mathfrak{T} = \langle U, \mathbb{I} \rangle$, where U is a non-empty set, the *universe* of the space, and $\mathbb{I}$ an *interior operator* on U.[3] Individual variables of **RCC** range over *non-empty regular closed sets* of $\mathfrak{T}$, i.e., an *assignment* in $\mathfrak{T}$ is a map $\mathfrak{a}$ associating with every variable X a set $\mathfrak{a}(X) \subseteq U$ such that $\mathfrak{a}(X) = \mathbb{C}\mathbb{I}\,\mathfrak{a}(X)$, where $\mathbb{C}(X) \;=_{def}\; -\mathbb{I}(-X)$ is the closure operator on U dual to $\mathbb{I}$. The connection relation $\mathsf{C}(X, Y)$ is then interpreted as meaning that the point sets denoted by X and

[3] This means that $\mathbb{I}$ must satisfy the axioms $\mathbb{I}(X) \subseteq X$, $\mathbb{I}\mathbb{I}(X) = \mathbb{I}(X)$, $\mathbb{I}(U) = U$ and $\mathbb{I}(X \cap Y) = \mathbb{I}(X) \cap \mathbb{I}(Y)$ (Kuratowski, 1933).

Y share at least one point:

$$\mathfrak{T} \models^{\mathfrak{a}} \mathsf{C}(X, Y) \quad \text{iff} \quad \mathfrak{a}(X) \cap \mathfrak{a}(Y) \neq \emptyset.$$

The full first-order theory of **RCC** is too expressive to be computationally useful and is in fact undecidable (this follows from (Grzegorczyk, 1951); results applying more specifically to **RCC** can be found in (Gotts, 1996b) and (Dornheim, 1998)). Fortunately, there are various decidable (and even tractable) fragments of **RCC**.

As was mentioned above, for certain applications one can limit the relations employed to the set **RCC-8**. Moreover, according to the experiments reported by Knauff et al. (1997) the eight predicates turn out to be 'cognitively adequate' in the sense that people indeed distinguish between those relations. Formally, the language of **RCC-8** consists of a set of individual variables $X_0, X_1, \ldots$, called *region variables*, the eight binary predicates DC, EC, PO, EQ, TPP, TPPi, NTPP, NTPPi, and the Boolean connectives ($\neg$, $\wedge$, $\vee$, $\rightarrow$), out of which we can construct *spatial formulas*.

RCC-8 is interpreted in topological spaces $\mathfrak{T} = \langle U, \mathbb{I} \rangle$: as before, the region variables range over non-empty regular closed sets in $\mathfrak{T}$, and the eight predicates are defined in the following way:

$$
\begin{array}{ll}
\mathsf{DC}(X_1, X_2) & \neg\exists x[x \in X_1 \cap X_2] \\
\mathsf{EC}(X_1, X_2) & \exists x[x \in (X_1 \cap X_2)] \wedge \neg\exists x[x \in (\mathbb{I}X_1 \cap \mathbb{I}X_2)] \\
\mathsf{PO}(X_1, X_2) & \exists x[x \in (\mathbb{I}X_1 \cap \mathbb{I}X_2)] \wedge \exists x[x \in (\mathbb{I}(X_1) \cap -X_2)] \\
 & \wedge \exists x[x \in (-X_1 \cap \mathbb{I}X_2)] \\
\mathsf{EQ}(X_1, X_2) & \forall x[x \in X_1 \leftrightarrow x \in X_2] \\
\mathsf{TPP}(X_1, X_2) & \forall x[x \in (-X_1 \cup X_2)] \wedge \exists x[x \in (X_1 \cap -\mathbb{I}X_2)] \\
 & \wedge \exists x[x \in (-X_1 \cap X_2)] \\
\mathsf{NTPP}(X_1, X_2) & \forall x[x \in (-X_1 \cup \mathbb{I}X_2)] \wedge \exists x[x \in (-X_1 \cap X_2)].
\end{array}
$$

For example, according to this definition, $\mathsf{EC}(X, Y)$ means that X and Y share at least one point but do not share any interior point (i.e. they only share boundary points).

The main reasoning task for **RCC-8** can be formulated as follows:

– given a finite set Σ of spatial formulas, decide whether Σ is satisfiable in a topological space, i.e., whether there exists a topological space $\mathfrak{T}$ and an assignment $\mathfrak{a}$ in it such that $\mathfrak{T} \models^{\mathfrak{a}} \Sigma$.

That this *satisfiability problem* is decidable was observed by Bennett (1994), who exploited the results of Tarski (1938) to encode **RCC-8** into intuitionistic propositional logic. Subsequently, Bennett (1996) gave another embedding of **RCC-8** into the more expressive logic containing Lewis's **S4** modality and an additional universal modality, whose meaning will be explained below. In the current paper we shall refer to this bi-modal logic as $\mathbf{S4_u}$. The embedding is based on the result of McKinsey and Tarski (1948) according to which **S4** is characterised by the class of topological spaces in the following sense:

Given a topological space $\mathfrak{T} = \langle U, \mathbb{I} \rangle$ and a modal formula φ, we interpret the propositional variables in φ as subsets of U, the Boolean connectives as the corresponding set-theoretic operations, the necessity operator $\boldsymbol{I}$ as $\mathbb{I}$, and the possibility operator $\boldsymbol{C}$ as $\mathbb{C}$ (we denote the usual box $\Box$ and diamond $\Diamond$ of **S4** by $\boldsymbol{I}$ and $\boldsymbol{C}$ to emphasize their topological meaning). Now, if the value of φ is the whole space U, no matter what sets are assigned to its variables and what topological space is taken, then φ is provable in **S4**, and *vice versa*.

According to (Goranko and Passy, 1992), this completeness theorem still holds if we extend **S4** with the universal modalities $\forall$ and $\exists$ interpreted in $\mathfrak{T}$ as 'for all points in $\mathfrak{T}$' and 'there is a point in $\mathfrak{T}$', respectively. That is the value of $\forall\varphi$ (under a certain interpretation) is U if the value of φ (under this interpretation) is U, otherwise it is $\emptyset$; $\exists\varphi \ \equiv_{def} \ \neg\forall\neg\varphi$.

It is easy to see that the language of the resulting logic $\mathbf{S4_u}$ is expressive enough to encode the topological meaning of the **RCC-8** formulas as defined above. Given such a formula φ, we replace in it ocurrences of **RCC-8** predicates with the corresponding modal formulas, e.g.

$$\mathsf{DC}(X_i, X_j) \text{ with } \neg\exists(p_i \wedge p_j),$$

$$\mathsf{NTPP}(X_i, X_j) \text{ with } \forall(\neg p_i \vee \boldsymbol{I}p_j) \wedge \exists(\neg p_i \wedge p_j),$$

$$\text{etc.}$$

(p_i, p_j are propositional variables) and add to the result the conjunct

$$\exists p_i \wedge (p_i \leftrightarrow \boldsymbol{C}\boldsymbol{I}p_i)$$

for every region variable X_i in φ to ensure that the propositional variables are interpreted by non-empty regular closed sets of topological spaces. The resulting modal formula is denoted by $\varphi^\dagger$.

Now, if we recall that some (in particular, all finite) topological spaces are determined by Kripke frames $\langle W, R\rangle$ for **S4** — every such frame gives rise to the topological space $\langle W, \mathbb{I}\rangle$, where

$$\mathbb{I}X = \{x \in X : \forall y \in W \ (xRy \to y \in X)\}$$

for any $X \subseteq W$—and that $\mathsf{S4}_u$ has the finite model property (Goranko and Passy, 1992), then we immediately obtain

Theorem 1. *For every* **RCC-8** *formula* φ*, the following conditions are equivalent:*

(i) φ *is satisfiable in a topological space,*

(ii) $\varphi^\dagger$ *is satisfiable in a topological space,*

(iii) $\varphi^\dagger$ *is satisfiable in some finite Kripke frame for* **S4***.*

Thus we reduce the satisfiability problem for **RCC-8** formulas to the satisfiability problem for propositional bimodal formulas in Kripke frames for $\mathbf{S4_u}$, which is decidable (Goranko and Passy, 1992).

Moreover, one can show that every satisfiable formula of the form $\varphi^\dagger$ can be satisfied in a partially ordered Kripke frame each point in which has at most two (incomparable) successors, and the number of worlds in this frame does not exceed the length $\ell(\varphi^\dagger)$ of $\varphi^\dagger$. [** Is the length the number of variable instances?? *] This result was obtained by Renz (1998); a somewhat more general theorem is proved in (Wolter and Zakharyaschev, 2000a). It follows in particular that the satisfiability problem for **RCC-8** formulas is NP-complete; see (Renz and Nebel, 1997; Renz and Nebel, 1999) where maximal tractable fragments of **RCC-8** are also described. It also follows that all satisfiable **RCC-8** formulas can be satisfied in $\mathbb{R}^n$ for any $n \geq 1$. (Note however that $\mathbf{S4_u}$ is not complete with respect to $\mathbb{R}^n$.)

3. Point-based temporal RCC-8

One approach to constructing spatio-temporal logics is to combine **RCC-8** with point-based temporal logics, for instance, the well-known *propositional temporal logic* **PTL** interpreted on the flow of time $\langle \mathbb{N}, < \rangle$ and having the temporal operators $\mathcal{S}$ (Since) and $\mathcal{U}$ (Until). Other standard operators can be defined:

- $\bigcirc\varphi \equiv_{def} \varphi\,\mathcal{U}\,\varphi$ ('At the next moment φ'),
- $\Diamond^+\varphi \equiv_{def} (p \vee \neg p)\,\mathcal{U}\,\varphi$ ('At some time in the future φ')
- $\Box^+\varphi \equiv_{def} \neg\Diamond^+\neg\varphi$ ('Always in the future φ'),

and similarly for their past counterparts $\bigcirc^-$, $\Box^-$ and $\Diamond^-$.

The spatial regions occupied by the objects under consideration may change with time passing by, but the topological space in which they are moving always remains the same. This naïve picture is formalised by the following concept of topological temporal model.

Definition 1. A *topological temporal model* (or *tt-model*, for short) based on a topological space $\mathfrak{T} = \langle U, \mathbb{I} \rangle$ is a triple of the form $\mathfrak{M} = \langle \mathfrak{T}, \mathbb{N}, \mathfrak{a} \rangle$, where $\mathfrak{a}$, an *assignment* in $\mathfrak{T}$, associates with every region variable X and every moment of time $n \in \mathbb{N}$ a non-empty regular closed subset of U. For each n, we take $\mathfrak{a}_n$ to be the function defined by $\mathfrak{a}_n(X) = \mathfrak{a}(X, n)$.

There are several different ways of introducing a temporal dimension into the syntax of RCC-8. The most obvious is to allow applications of the operators $\mathcal{S}$ and $\mathcal{U}$ (along with the Booleans) to spatial formulas. We call the resulting spatio-temporal language $\mathbf{ST}_0$.

Definition 2. For a tt-model $\mathfrak{M} = \langle \mathfrak{T}, \mathbb{N}, \mathfrak{a} \rangle$, an $\mathbf{ST}_0$-formula φ, and $n \in \mathbb{N}$, define the *truth-relation* $(\mathfrak{M}, n) \models \varphi$, meaning '$\varphi$ holds in $\mathfrak{M}$ at moment n,' by induction on the construction of φ:

- if φ contains no temporal operators, then $(\mathfrak{M}, n) \models \varphi$ iff $\mathfrak{T} \models^{\mathfrak{a}_n} \varphi$;
- $(\mathfrak{M}, n) \models \varphi \,\mathcal{U}\, \psi$ iff there is $k > n$ such that
  $$(\mathfrak{M}, k) \models \psi \text{ and } (\mathfrak{M}, l) \models \varphi \text{ for every } l \text{ such that } n < l < k;$$
- $(\mathfrak{M}, n) \models \varphi \,\mathcal{S}\, \psi$ iff there is $k < n$ such that
  $$(\mathfrak{M}, k) \models \psi \text{ and } (\mathfrak{M}, l) \models \varphi \text{ for every } l \text{ such that } k < l < n.$$

$\mathbf{ST}_0$ allows us to say many things about changes of spatial relationships over time. For example, $\neg\Box^+\mathsf{P}(\mathit{Kosovo}, \mathit{Yugoslavia})$ means that 'Kosovo will not always be part of Yugoslavia.' It is also expressive enough to constrain movements to be continuous, in so far as one can describe possible continuous transitions among the **RCC-8** relations that can hold between two regions (such transitions were identified by Randell et al. (1992a) and the importance has been emphasised more recently by Muller (1998)):

$$\Box^+(\mathsf{DC}(X,Y) \to \bigcirc(\mathsf{DC}(X,Y) \vee \mathsf{EC}(X,Y))),$$
$$\Box^+(\mathsf{EC}(X,Y) \to \bigcirc(\mathsf{DC}(X,Y) \vee \mathsf{EC}(X,Y) \vee \mathsf{PO}(X,Y))),$$
$$\Box^+(\mathsf{PO}(X,Y) \to \bigcirc(\mathsf{EC}(X,Y) \vee \mathsf{PO}(X,Y) \vee$$
$$\mathsf{TPP}(X,Y) \vee \mathsf{EQ}(X,Y) \vee \mathsf{TPP}^{-1}(X,Y))),$$

etc.

However, the expressive power of $\mathbf{ST}_0$ is limited in that one can only employ the **RCC-8** relations to compare region variables at the same point in time; so it does not allow one to describe spatial relations between the extensions of a region variable at different times. For example, there is no way to say something like 'The extension of the EU is a part of what its extension will be next year (i.e. at the next time point).'

To overcome this limitation we introduce the logic $\mathbf{ST}_1$, which extends $\mathbf{ST}_0$ by allowing applications of the next-time operator $\bigcirc$ not only to formulas but also to region variables. Thus, arguments of the RCC-8 predicates are now *region terms*, which consist of a region variable that may be prefixed by an arbitrarily long sequence of $\bigcirc$ operators. For instance $\bigcirc\bigcirc EU$ could denote the region occupied by the EU in two years time and one can write formulas such as $\mathsf{P}(EU, \bigcirc\bigcirc EU)$. The semantics of $\mathbf{ST}_1$ is just that of $\mathbf{ST}_0$ extended by the following clause:

- $\mathfrak{a}(\bigcirc X, n) = \mathfrak{a}(X, n+1)$.

Using $\mathbf{ST}_1$ we can now talk about the changing extensions of individual region variables. For instance $\Box^+\mathsf{P}(EU, \bigcirc EU)$ says that 'the EU will never shrink'. The new construct may also be used to refine the continuity assumption by requiring that $\Box^+(\mathsf{EQ}(X, \bigcirc X) \vee \mathsf{O}(X, \bigcirc X))$, i.e., 'regions X and $\bigcirc X$ either coincide or overlap.' We can also express the condition that the extension of a region variable X is fixed for the future: $\Box^+\mathsf{EQ}(X, \bigcirc X)$; or that it has at most two distinct states, one on even days, another on odd ones: $\Box^+\mathsf{EQ}(X, \bigcirc\bigcirc X)$. Note, that the $\mathbf{ST}_1$-formula $\Box^+\mathsf{NTPP}(X, \bigcirc X)$ is satisfiable only in models based on infinite topological spaces, whereas for formulas of $\mathbf{ST}_0$ (and of course RCC-8) finite topological spaces always suffice.

It may appear that $\mathbf{ST}_1$ can compare regions that are separated in time only by limited sequences of time points. However, using an auxiliary variable, whose extension is constrained to be constant over time, we can write, for instance, $\Box^+\mathsf{EQ}(X, \bigcirc X) \wedge \Diamond^+\mathsf{EQ}(X, EU) \wedge \mathsf{P}(Russia, X)$, which is satisfiable iff 'someday in the future the *present* territory of Russia will be part of the EU.' This contrasts with the formula $\Diamond^+\mathsf{P}(Russia, EU)$ meaning that there will be a day when Russia (its territory on that day — perhaps without Chechnya but with Byellorussia) becomes part of the EU.

Imagine now that we want to say that every location in Europe will pass through the Euro-zone, but only the land currently occupied by Germany will use the Euro forever. Unfortunately, we do not know which countries will form within Europe in the future, so we can't simply write down all formulas of the form $\Diamond^+\mathsf{P}(X, \textit{Euro-zone})$. What we need is the possibility of constructing regions $\Diamond^+X$ containing all the points that will belong to region X at some time in the future and $\Box^+X$ containing just those points included in all future states of X. Then we can write: $\mathsf{P}(Europe, \Diamond^+\textit{Euro-zone})$ and $\mathsf{EQ}(Germany, \Box^+\textit{Euro-zone})$. Similarly, the formula $\mathsf{P}(Russia, \Diamond^+EU)$ says that all points of the present territory of Russia will belong to the EU at some time in the future (but perhaps at different moments of time).

To permit fully general application of temporal operators to region variables we define the language $\mathbf{ST}_2$ in which region variables may be prefixed by arbitrary srtings of the operators $\bigcirc$, $\Box^+$, $\Diamond^+$. The semantics of the new operators are given by:

- $\mathfrak{a}(\Box^+t, n) = \mathbb{CI}\left(\bigcap_{k>n} \mathfrak{a}(t, k)\right)$,
- $\mathfrak{a}(\Diamond^+t, n) = \mathbb{CI}\left(\bigcup_{k>n} \mathfrak{a}(t, k)\right)$.

Finally, we can make our languages $\mathbf{ST}_i$, for $i = 0, 1, 2$, even more expressive by allowing applications of the Boolean operations to region terms. Their semantical meaning is defined as follows: $\mathfrak{a}(t \vee t', n) = \mathbb{CI}(\mathfrak{a}(t, n) \cup \mathfrak{a}(t', n))$, $\mathfrak{a}(t \wedge t', n) = \mathbb{CI}(\mathfrak{a}(t, n) \cap \mathfrak{a}(t', n))$, $\mathfrak{a}(\neg t, n) = \mathbb{CI}(U - \mathfrak{a}(t, n))$. Let $\mathbf{ST}_i^+$ be the resulting family of languages. In these languages we can write formulas such as $\mathsf{EQ}(\mathit{UK}, \mathit{Great\_Britain} \vee \mathit{Northern\_Ireland})$, meaning that the extension of the UK is the sum of the extensions of Great Britain and Northern Ireland.

4. Modal Encoding of $\mathbf{ST}_2^+$

The decidability and complexity results for the $\mathbf{ST}_i$ languages were proved in (Wolter and Zakharyaschev, 2000b) by embedding them into the two-dimensional propositional modal logic $\mathbf{S4_u} \times \mathbf{PTL}$, the Cartesian product of the logics $\mathbf{S4_u}$ and $\mathbf{PTL}$. We call this logic *Propositional Spatio-Temporal Logic* or **PSTL**. Its connectives are: the Booleans, the necessity and possibility operators $\boldsymbol{I}$ and $\boldsymbol{C}$ of **S4**, the universal necessity and possibility operators $\forall$ and $\exists$, and the temporal operators $\mathcal{S}$ and $\mathcal{U}$. It is not known whether the full **PSTL** (or even $\mathbf{S4} \times \mathbf{PTL}$) is decidable. However, its fragments corresponding to the $\mathbf{ST}_i$ languages are quite manageable.

The modal translation from $\mathbf{ST}_2^+$ into **PSTL** is defined by extending the † transformation specified in Section 2 to handle region terms containing Boolean and temporal operators. The region terms are already ostensibly formulas of **PSTL**; however, we must bear in mind that these terms are intended to denote only regular closed subsets of the space. The easiest way to take this into account is to prefix $\boldsymbol{CI}$ to every subterm occuring in an $\mathbf{ST}_2^+$-formula.

What is the intended semantics of **PSTL**? As we have seen, when encoding pure **RCC-8** into $\mathbf{S4_u}$ we can specify the semantics with either topological models or Kripke models and these are equivalent in terms of validity. However, the addition of the temporal component makes the situation more complicated in that the topological and Kripke semantics do not perfectly agree on the class of valid formulas. Let us define these two types of model structures:

Definition 3. A *Kripke* **PSTL**-*model* is a triple $\mathfrak{K} = \langle \mathfrak{F}, \mathbb{N}, \mathfrak{V} \rangle$ where $\mathfrak{F} = \langle W, R \rangle$ is a quasi-order (a frame for **S4**) and $\mathfrak{V}$, a *valuation*, is a map associating with every propositional variable p and every $n \in \mathbb{N}$ a subset $\mathfrak{V}(p, n) \subseteq W$. For each spatial point $u \in W$ and each time point $n \in \mathbb{N}$, the truth-relation $(u, n) \models_{\mathfrak{K}} \varphi$ is defined by:

- $(u, n) \models_{\mathfrak{K}} p$ iff $u \in \mathfrak{V}(p, n)$,
- $(u, n) \models_{\mathfrak{K}} \forall \psi$ iff $(v, n) \models_{\mathfrak{K}} \psi$ for all $v \in W$,
- $(u, n) \models_{\mathfrak{K}} \boldsymbol{I} \psi$ iff $(v, n) \models_{\mathfrak{K}} \psi$ for all $v \in W$ such that uRv,
- $(u, n) \models_{\mathfrak{K}} \psi \, \mathcal{U} \, \chi$ iff there is $k > n$ such that
$$(u, k) \models_{\mathfrak{K}} \chi \text{ and } (u, m) \models_{\mathfrak{K}} \psi \text{ for all } m,\ n < m < k,$$

plus the standard clauses for $\mathcal{S}$ and the Booleans. A **PSTL**-formula φ is *satisfied* in $\mathfrak{K}$ if $(u, n) \models_{\mathfrak{K}} \varphi$ for some $u \in W$ and $n \in \mathbb{N}$.

Definition 4. A *topological* **PSTL**-*model* is a structure $\mathfrak{N} = \langle \mathfrak{T}, \mathbb{N}, \mathfrak{U} \rangle$ in which $\mathfrak{T} = \langle U, \mathbb{I} \rangle$ is a topological space and $\mathfrak{U}$ is a map associating with every propositional variable p and every $n \in \mathbb{N}$ a set $\mathfrak{U}(p, n) \subseteq U$. $\mathfrak{U}$ is then extended to arbitrary **PSTL**-formulas in the following way:

- $\mathfrak{U}(\psi \wedge \chi, n) = \mathfrak{U}(\psi, n) \cap \mathfrak{U}(\chi, n)$,
- $\mathfrak{U}(\neg\psi, n) = U - \mathfrak{U}(\psi, n)$,
- $\mathfrak{U}(\forall\psi, n) = U$ if $\mathfrak{U}(\psi, n) = U$, and $\mathfrak{U}(\forall\psi, n) = \emptyset$ otherwise,
- $\mathfrak{U}(\boldsymbol{I}\psi, n) = \mathbb{I}\mathfrak{U}(\psi, n)$,
- $x \in \mathfrak{U}(\psi\, \mathcal{U}\, \chi, n)$ iff there is a $k > n$ such that $x \in \mathfrak{U}(\chi, k)$ and $x \in \mathfrak{U}(\psi, m)$ for all $m, n < m < k$.

Consequently the defined temporal operators are interpreted by

- $\mathfrak{U}(\bigcirc\, \psi, n) = \mathfrak{U}(\bigcirc\, \psi, n+1)$,
- $\mathfrak{U}(\Diamond^{+} \psi, n) = \bigcup_{k>n} \mathfrak{U}(\psi, k)$,
- $\mathfrak{U}(\Box^{+} \psi, n) = \bigcap_{k>n} \mathfrak{U}(\psi, k)$.

A **PSTL**-formula φ is *satisfied* in $\mathfrak{N}$ if $\mathfrak{U}(\varphi, n) \neq \emptyset$ for some $n \in \mathbb{N}$.

The sets of **PSTL**-formulas satisfiable in Kripke models and topological models turn out to be different. Of course, every Kripke model is equivalent to some topological model. But the converse does not hold. A good example is provided by the formula $\Diamond^{+}\boldsymbol{C}p \leftrightarrow \boldsymbol{C}\Diamond^{+}p$. This is valid in every Kripke **PSTL**-model because from any space-time point in such a structure any given other point is reachable by forward transition along the time line followed by transition along the accessibility relation of $\mathfrak{F}$ just in case it is reachable by first moving along the accessibility relation and then forward along the time dimension (since the accessibilty relation remains constant for all time points). However, in certain infinite topological spaces (e.g. $\mathbb{R}$) one can construct infinite sequences X_n of closed sets such that $\bigcup_{n\in\mathbb{N}} X_n$ is not closed. In a topological model based on such a space $\Diamond^{+}\boldsymbol{C}p$ need not have the same denotation as $\boldsymbol{C}\Diamond^{+}p$. This subtlety concerning infinite unions is not accounted for by the Kripke approach.

The divergence between topological and Kripke **PSTL**-models is problematic because whereas the topological models correspond to the desired spatial interpretation, most currently known methods of determining decidability and complexity are based on Kripke models. Fortunately, Wolter and Zakharyaschev (2000b) have been able to show that the two semantics agree on the satisfiability of $\mathbf{ST}_2^{+}$-formulas as long as we adopt a reasonably natural *Finite State Assumption* (FSA). The FSA requires that over the infinite sequence of time points each region that one can refer to can have only finitely many distinct extensions (but it may change its extension infinitely often). Although this restriction rules out many mathematically interesting possibilities, it is perfectly satisfactory for a wide range of practical applications (for example planning tasks where we want to get from an initial to a final situation — in a finite number of steps). To formalise the FSA we define the following restricted class of tt-models:

Definition 5. Say that a tt-model $\mathfrak{M} = \langle \mathfrak{T}, \mathbb{N}, \mathfrak{a} \rangle$ satisfies FSA, or is an *FSA-tt-model*, if for every region term t there are finitely many regular closed sets $A_1, \ldots, A_m \subseteq U$ such that $\{\mathfrak{a}(t, n) : n \in \mathbb{N}\} = \{A_1, \ldots, A_m\}$.

It can be shown that an $\mathbf{ST}_2^{+}$-formula is satisfiable in an FSA-tt-model iff it is satisfiable in an FSA-model based on a finite topological space.

Because of the very restricted combinations of operators that result from translating **RCC-8** predicates and their region term arguments, the modal translations of $\mathbf{ST}_2^{+}$-formulas form a rather special fragment of the modal language **PSTL**. As was mentioned above, Renz (1998) showed that an **RCC-8** formula φ is satisfiable iff $\varphi^\dagger$ is satisfiable in a Kripke model based on an **S4**-frame of depth ≤ 1 and width ≤ 2 (which means that it contains no chains of more than 2 distinct points, and no point has more than 2 distinct successors); and it turns out that this result can be generalised to formulas of $\mathbf{ST}_1^+$ and $\mathbf{ST}_2^+$:

Theorem 2.
*a) An $\mathbf{ST}_2^+$-formula φ is satisfiable in a FSA-tt-model iff $\varphi^\dagger$ is satisfiable in a Kripke **PSTL**-model (which also satisfies FSA) whose underlying **S4**-frame is of depth ≤ 1 and width ≤ 2.*[4]
*b) An $\mathbf{ST}_1^+$-formula φ is satisfiable in a tt-model iff $\varphi^\dagger$ is satisfiable in a Kripke **PSTL**-model of depth ≤ 1 and width ≤ 2.*

This result makes it possible to use the method of quasi-models (Wolter and Zakharyaschev, 1999) to prove that the satisfiability problem for all the languages $\mathbf{ST}_i^+$, $i = 1, 2$, in tt-models is decidable. Given the rather weak interaction between time and space in $\mathbf{ST}_0$ it is not hard to show that the satisfiability problem for $\mathbf{ST}_0$ formulas in tt-models is PSPACE-complete (the same as that of **PTL**). The satisfiability problem for $\mathbf{ST}_1$ formulas in tt-models is decidable in EXPSPACE (and becomes NP-complete for the sublanguage of $\mathbf{ST}_1$ with only the temporal operator $\bigcirc$). If we restrict the admissible models to FSA-models then the satisfiability problem for $\mathbf{ST}_2$ formulas is decidable in EXPSPACE. Moreover, all the complexity results just given remain valid after replacing $\mathbf{ST}_i$ with $\mathbf{ST}_i^+$.

[4] Actually, since occurences of $\Box^+$ within the region terms of $\mathbf{ST}_2^+$ are always regularised by adding the prefix $\boldsymbol{CI}$, one might conjecture that the FSA is not necessary. However, at present we cannot envisage how to prove this.

The topological temporal models we were considering above are based on the discrete flow of time $\mathbb{N}$. By replacing $\mathbb{N}$ with $\mathbb{Q}$, $\mathbb{R}$, or any other strict linear order we can extend our semantics to cover different flows of time. But then the question arises as to whether the decidability and complexity results proved for $\mathbb{N}$ can be extended to, say, the logic determined by the class of arbitrary strict linear orders, by the reals $\mathbb{R}$, or by the rationals $\mathbb{Q}$. It turns out that for the language of $\mathbf{ST}_0$ we can easily extend the decidability proof for $\mathbb{N}$ by using the fact that the propositional temporal logics based on those flows of time are decidable (Gabbay et al., 1994). As concerns $\mathbf{ST}_1$, observe that the operator $\bigcirc$ is meaningless for dense linear orders—thus for $\mathbb{Q}$ and $\mathbb{R}$ this language reduces to $\mathbf{ST}_0$. Decidability of $\mathbf{ST}_1$ interpreted in arbitrary linear orders is an open problem. And nothing is known about the decidability of $\mathbf{ST}_2$ interpreted in flows of time different from $\mathbb{N}$. We conjecture, however, that the methods developed in (Hodkinson et al., 2000) can be used to prove the decidability of $\mathbf{ST}_2$ based on any of the flows of time mentioned above (under FSA).

5. RCC-8 and Interval Temporal Logic

We have seen how the **RCC-8** theory can be temporalised by combining it with a point-based temporal logic. However, since the region-based approach to spatial reasoning was inspired by and closely mirrors the interval-based approach to temporal reasoning (Allen, 1981; Allen, 1984) (they both take extended entities, rather than points as primitives) it would seem far more natural to temporalise **RCC-8** by combining it with an interval-based logic. In this section we show that this can indeed be done; and moreover, that the resulting system can in fact be embedded by suitable syntactic definitions within the point-based logic **PSTL** defined above.

We remind the reader that Allen's logic has thirteen basic relations between time intervals — $\mathsf{Before}(i,j)$, $\mathsf{Meets}(i,j)$, $\mathsf{Overlaps}(i,j)$, $\mathsf{During}(i,j)$, $\mathsf{Starts}(i,j)$, $\mathsf{Finishes}(i,j)$, their inverses, and $\mathsf{Equal}(i,j)$. The set of formulas constructed using these predicates and the Booleans can be regarded as a temporal 'twin' of **RCC-8**.

Following Allen (1984), we write $\mathsf{HOLDS}(R,i)$ to say that the relation R holds during some time interval i. Thus $\mathsf{HOLDS}(\mathsf{PO}(X,Y),i)$ means that during interval i regions X and Y partially overlap. Let us call an **ARCC-8** *formula* any Boolean combination of basic temporal predicates and formulas of the form $\mathsf{HOLDS}(\varphi,i)$ where φ is an **RCC-8** formula. Here is a simple example of a valid entailment in this unsophisticated language:

$$\begin{array}{l}
\mathsf{Meets}(i,j),\ \mathsf{During}(i,k),\ \mathsf{During}(j,k),\\
\mathsf{HOLDS}(\mathsf{TPP}(\mathit{Hong_Kong},\mathit{UK}) \wedge \mathsf{EC}(\mathit{Hong_Kong},\mathit{China}),i),\\
\mathsf{HOLDS}(\mathsf{DC}(\mathit{Hong_Kong},\mathit{UK}),j),\\
\mathsf{HOLDS}(\mathsf{EC}(\mathit{UK},\mathit{China}) \vee \mathsf{DC}(\mathit{UK},\mathit{China}),k)\\
\ \models\quad \mathsf{HOLDS}(\mathsf{EC}(\mathit{UK},\mathit{China}),i)\ .
\end{array}$$

There are different views on how intervals should be modelled in different time flows. A common interpretation is that the intervals are treated as ordered pairs of distinct points of the domain $\mathbb{Q}$ or $\mathbb{R}$. Within our framework we can in fact adopt any of these models; but, for simplicity of presentation it is convenient to employ a semantics where intervals are arbitrary convex non-empty subsets of the time points of an arbitrary time flow. Thus we give the following semantics for **ARCC-8**:

Definition 6. An *it-model* (interval topological model) is the triple $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{T}, \mathfrak{a} \rangle$, where: $\mathfrak{F} = \langle W, < \rangle$ is a strict linear order (modelling the intended flow of time); $\mathfrak{T} = \langle U, \mathbb{I} \rangle$ is a topological space; and assignment $\mathfrak{a}$ associates with every interval variable i a non-empty convex subset of W and with every region variable X and every moment of time u it associates a regular closed set $\mathfrak{a}(X, u)$ in $\mathfrak{T}$.

The truth-relation is defined inductively as follows (clauses for the Booleans are standard):

- $\mathfrak{M} \models_{\mathfrak{a}} \mathsf{Before}(i, j)$ iff
$$(\exists t \in W)(\forall t_i t_j)[(t_i \in \mathfrak{a}(i) \wedge t_j \in \mathfrak{a}(j)) \rightarrow t_i < t < t_j];$$
- $\mathfrak{M} \models_{\mathfrak{a}} \mathsf{Meets}(i, j)$ iff $\quad (\forall t_i t_j)[(t_i \in \mathfrak{a}(i) \wedge t_j \in \mathfrak{a}(j)) \rightarrow t_i < t_j]$ and not $\mathfrak{M} \models_{\mathfrak{a}} \mathsf{Before}(i, j)$;
- $\mathfrak{M} \models_{\mathfrak{a}} \mathsf{Overlaps}(i, j)$ iff $\quad (\exists t)[t \in \mathfrak{a}(i) \cap \mathfrak{a}(j)] \wedge$ $(\exists t_i \in \mathfrak{a}(i))(\forall t_j \in \mathfrak{a}(j))[t_i < t_j] \wedge (\exists t_j \in \mathfrak{a}(j))(\forall t_i \in \mathfrak{a}(i))[t_i < t_j]];$
- The other interval relations are defined similarly;
- $\mathfrak{M} \models_{\mathfrak{a}} \mathsf{HOLDS}(\varphi, i)$ iff for every point $u \in \mathfrak{a}(i)$ we have $\mathfrak{T} \models_{\mathfrak{a}_u} \varphi$.

We now show how by using some ideas of Blackburn (1992) the language **ARCC-8** can be directly embedded into **PSTL**. For convenience we define undirected universal and existential modalities over time: $\Box\varphi = \varphi \wedge \Box^{+}\varphi \wedge \Box^{-}\varphi$ (i.e., φ is true at every moment) and $\Diamond\varphi = \neg\Box\neg\varphi$. We extend the translation function † defined above to encode the interval relations and HOLDS predicate. For example, we replace in φ

$$\begin{aligned}
\mathsf{Equals}(i,j) &\text{ with } \Box(t_i \leftrightarrow t_j)\\
\mathsf{Before}(i,j) &\text{ with } \Box(t_i \rightarrow (\neg t_j \wedge \Diamond^+ t_j)) \wedge \Diamond(\neg t_i \wedge \neg t_j \wedge \Diamond^- t_i \wedge \Diamond^+ t_j)\\
\mathsf{Meets}(i,j) &\text{ with } \Box(t_i \rightarrow \neg t_j \wedge \Diamond^+ t_j) \wedge \neg\Diamond(\neg t_i \wedge \neg t_j \wedge \Diamond^- t_i \wedge \Diamond^+ t_j)\\
\mathsf{Overlaps}(i,j) &\text{ with } \Diamond(t_i \wedge t_j) \wedge \Diamond(t_i \wedge \neg t_j \wedge \Diamond^- t_j) \wedge \Diamond(t_j \wedge \neg t_i \wedge \Diamond^+ t_i)\\
\mathsf{HOLDS}(\varphi, i)) &\text{ with } \Box(t_i \rightarrow \varphi^\dagger)
\end{aligned}$$

and the other Allen relations can be encoded in similar fashion. Here the variables t_i are just ordinary propositional variables; however, we need to constrain their interpretation so that they can be employed to stand for intervals. Therfore, we add to the resulting formula the conjuncts

$$\Diamond t_i \wedge \Box(\Diamond^- t_i \wedge \Diamond^+ t_i \rightarrow t_i) \wedge (t_i \rightarrow \forall t_i)$$

for every interval variable i occurring in φ, thus obtaining $\varphi^\dagger$. The first conjunct ensures that t_i is non-empty;[5] the second that it is a convex interval of the time series; and the final conjunct means that the value of t_i is constant relative to the spatial dimension.

It can be shown that an **ARCC-8** formula φ is satisfiable in an it-model just in case its modal translation $\varphi^\dagger$ is satisfiable in a Kripke **PSTL**-model. Moreover, the satisfiability problem for **ARCC-8** formulas in it-models is NP-complete. Thus **ARCC-8** has the same computational complexity as both **RCC-8** and the constraint language of Allen's temporal interval predicates (Vilain et al., 1986).

[5] If we replace this conjunct by $\Diamond(t_i \wedge \Diamond^+ t_i)$ we can also rule out 'intervals' consisting of a single time point.

6. How Far Can Multi-Dimensional Modal Logic Take Us?

We have seen how the 2-dimensional language **PSTL** provides an expressive formalism for representing spatio-temporal information, which encompasses both topological constraints and linear temporal logic. We now consider what other concepts one might want to represent and note some difficulties that arise due to known complexity results.

A concept that is very useful for describing real situations is that of spatial *convexity*. The addition to the first-order **RCC** formalism of a function giving the 'convex-hull' of a region proposed in (Randell et al., 1992a) (such a function and a convexity predicate true of convex regions are interdefinable). First-order languages including topological relations and a convexity predicate have been found to be highly expressive (Pratt, 1999). Although satisfiability of a combination of **RCC-8** relations and convexity predicates is known to be decidable is has also been shown to be as hard as solving systems of non-linear constraints over $\mathbb{R}$ (Davis et al., 1999). It is unclear whether modal logics can contribute to reasoning about convexity. The obvious models of multi-dimensional space within which convexity constraints could be specified involve cross products of linearly ordered infinite frames corresponding to coordinate axes of the space. However, it is known that all modal logics of such products are undecidable (Reynolds and Zakharyaschev, 2000). Balbiani et al. (1997) have given a modal logic of incidence geometry within which one can express collinearity and betweenness (and hence convexity) but it is not known whether this logic is decidable.

Another concept crucial to real situation descriptions is that of *connectedness* of regions. The objects we normally think and talk about are connected in the sense that any two points within an object can be joined by a path that lies entirely within the object. One of the most important open problems in spatial reasoning is whether there is a decision procedure for testing satisfiability of sets of topological relations holding among *connected planar* regions. A lower exponential complexity bound follows from the results of (Kratochvíl, 1991; Kratochvíl and Matoušek, 1991) concerning an analogous problem for planar graphs; and this result applies even in the case where we deal with only the basic non-disjunctive relations of **RCC-8**. We speculate that a modal analysis might shed some light on this decision problem.

One might also want to introduce more expressive power in the temporal dimension. In order to describe continuous changes more adequately one may want to employ a logic based on a dense model of time (Barringer et al., 1986). Within such a time flow, we must consider whether spatial relationships hold over open or closed intervals. This problem has be examined by Galton (1997,2000), who argues that whether a spatial relation is true at the bounding point of an interval over which it holds depends on the nature of the relation in question. This analysis may allow for more detailed description of changing spatial relationships than is possible within Allen's (1984) treatment. Also, rather than treating time as a linear ordering one might like to model alternative possible histories in terms of some branching structure, such as is described by the logic $\mathbf{CTL}^*$ (Emerson and Halpern, 1986). Semantically it is straightforward to combine the spatial interpretation of $\mathbf{S4_u}$ with any reasonable temporal logic; but so far nothing is known about the complexity of combinations with non-linear time-flows.

In considering decidability and complexity of multi-modal systems, logicians have almost always looked at the problem for arbitrary formulas of the resulting modal language. However, as we saw in the case of **PSTL** sub-languages with severely restricted syntax (e.g. $\mathbf{ST}_i^{(+)}$) may be able to express a significant vocabulary sufficient for many applications. Thus we suggest that looking for such sub-languages is a research area of great potential.

In order to really demonstrate the utility of multi-dimensional logics one would need to develop and implement reasoning algorithms capable of carrying out useful reasoning tasks. Automated reasoning with multi-modal logics is still in its infancy; but recently there have been some successes in developing proof methods (Dixon et al., 1998; Hustadt et al., 2000). A tableau calculus for the *local cubic* logic $\mathbf{LC}_2$, which is closely related to $\mathbf{S5}\times\mathbf{S5}$, has been implemented by Marx et al. (1999).

7. Conclusion

We have outlined the general structure of a knowledge representation formalism, based on multi-dimensional modal logics. This framework seems to be well suited for representing a large vocabulary of useful high-level spatio-temporal relations. Our decidability and complexity results for the languages $\mathbf{ST}_i^{(+)}$ and **ARCC-8** show that this approach enables one to construct very expressive yet effective spatio-temporal languages. We hope that the reader will take from this paper not only the particular details of the language **PSTL** but a wider appreciation of the possibilities of applying multi-dimensional modal logics to the development of Artificial Intelligence. We would also like to suggest that spatio-temporal reasoning is an area within which cooperation between pure logicians and researchers tackling specific reasoning problems arising in applications can lead to both interesting theorems and powerful practical algorithms.

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