

# Living Without Beth and Craig: Definitions and Interpolants in Description Logics with Nominals and Role Inclusions

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**Abstract.** The Craig interpolation property (CIP) states that an interpolant for an implication exists iff it is valid. The projective Beth definability property (PBDP) states that an explicit definition exists iff a formula stating implicit definability is valid. Thus, the CIP and PBDP transform potentially hard existence problems into deduction problems in the underlying logic. Description Logics with nominals and/or role inclusions do not enjoy the CIP nor PBDP, but interpolants and explicit definitions have many potential applications in ontology engineering and ontology-based data management. In this article we show the following: even without Craig and Beth, the existence of interpolants and explicit definitions is decidable in description logics with nominals and/or role inclusions such as  $\mathcal{ALCO}$ ,  $\mathcal{ALCH}$  and  $\mathcal{ALCHIO}$ . However, living without Craig and Beth makes this problem harder than deduction: we prove that the existence problems become  $2\text{EXPTIME}$ -complete, thus one exponential harder than validity. The existence of explicit definitions is  $2\text{EXPTIME}$ -hard even if one asks for a definition of a nominal using any symbol distinct from that nominal, but it becomes  $\text{EXPTIME}$ -complete if one asks for a definition of a concept name using any symbol distinct from that concept name.

## 1 Introduction

The *Craig Interpolation Property* (CIP) for a logic  $\mathcal{L}$  states that an implication  $\varphi \Rightarrow \psi$  is valid in  $\mathcal{L}$  iff there exists a formula  $\chi$  in  $\mathcal{L}$  using only the common symbols of  $\varphi$  and  $\psi$  such that  $\varphi \Rightarrow \chi$  and  $\chi \Rightarrow \psi$  are both valid in  $\mathcal{L}$ .  $\chi$  is then called an  $\mathcal{L}$ -interpolant for  $\varphi \Rightarrow \psi$ . The CIP is generally regarded as one of the most important and useful properties in formal logic [51], with numerous applications ranging from formal verification [43], to theory combinations [16, 19, 9] and query reformulation and rewriting in databases [42, 48, 6]. Description logics (DLs) are no exception [12, 47, 32, 13, 39, 26]. A particularly important consequence of the CIP in DLs is the *projective Beth definability property* (PBDP), which states that a concept is implicitly definable using a signature  $\Sigma$  of symbols iff it is explicitly definable using  $\Sigma$ . If the concept is a concept name and  $\Sigma$  the set of all

symbols distinct from that concept name, then we speak of the (non-projective) Beth definability property (BDP).

The BDP and PBDP have been used in ontology engineering to extract equivalent acyclic terminologies from ontologies [12, 13], they have been investigated in ontology-based data management to equivalently rewrite ontology-mediated queries [47, 50], and they have been proposed to support the construction of alignments between ontologies [26]. The CIP is often used as a tool to compute explicit definitions [12, 13, 50]. It is also the basic logical property that ensures the robust behaviour of ontology modules [31]. In the form of parallel interpolation it has been investigated in [32] to decompose ontologies. In [39], it is used to study P/NP dichotomies in ontology-based query answering. The PBDP is also related to the computation of referring expressions in linguistics [35] and in ontology-based data management [7]. It has been convincingly argued [8] that very often in applications the individual names used in ontologies are insufficient “to allow humans to figure out what real-world objects they refer to.” A natural way to address this problem is to check for such an individual name  $a$  whether there exists a concept  $C$  not using  $a$  that provides an explicit definition of  $a$  under the ontology  $\mathcal{O}$  and present such a concept  $C$  to the human user. Also very recently, it has been observed that strongly separating concepts for positive and negative examples given as data items in a knowledge base can be represented as interpolants, for appropriately defined ontologies and implications [18, 27, 28]. Thus, under the approach to DL concept learning proposed in [17, 36], searching for a solution to the concept learning problem can be reduced to computing an interpolant.

The CIP, PBDP, and BDP are so powerful because intuitively very hard existence questions are reduced to straightforward deduction questions: an interpolant *exists* iff an implication is valid and an explicit definition *exists* iff a straightforward formula stating implicit definability is valid. The existence problems are thus not harder than validity. For example, in the DL  $\mathcal{ALC}$ , the existence of an interpolant or an explicit definition can be decided in EXPTIME simply because deduction in  $\mathcal{ALC}$  is in EXPTIME (and without ontology even in PSPACE).

Unfortunately, the CIP and the PBDP fail to hold for many standard DLs. Particularly important examples of failure are the extension  $\mathcal{ALCO}$  of  $\mathcal{ALC}$  with nominals, the extension  $\mathcal{ALCH}$  of  $\mathcal{ALC}$  with role inclusions, and extensions of these with inverse roles and the universal role. To illustrate, even for very simple implications such as  $(\{a\} \sqcap \exists r.\{a\}) \sqsubseteq (\{b\} \rightarrow \exists r.\{b\})$  no  $\mathcal{ALCO}$ -interpolant exists. Moreover, at least for nominals, there is no satisfactory way to extend the expressive power of (expressive) DLs with nominals to ensure the existence of interpolants as validity is undecidable in any extension of  $\mathcal{ALCO}$  with the CIP [10].

The aim of this paper is to investigate the complexity of deciding the existence of interpolants and explicit definitions for DLs in which this cannot be deduced using the CIP or PBDP. We consider  $\mathcal{ALCO}$  and  $\mathcal{ALCH}$  and their extensions by inverse roles and/or the universal role. We note that both role inclusions and

nominals are part of the OWL 2 DL standard and are used in many real-world ontologies [53]. We prove that the existence of interpolants and the existence of explicit definitions are both  $2\text{EXPTIME}$ -complete in all cases, thus confirming the suspicion that these are harder problems than deduction if one has to live without Beth and Craig. For DLs with nominals, the  $2\text{EXPTIME}$  lower bound even holds if one asks for an explicit definition of a nominal over the signature containing all symbols distinct from that nominal, a scenario that is of particular interest for the study of referring expressions.

For the BDP, the situation is different as  $\mathcal{ALCH}$  and its extensions without nominals enjoy the BDP [11, 12]. Moreover, while  $\mathcal{ALCO}$  and  $\mathcal{ALCHO}$  do not enjoy the BDP [11, 12], we show here that their extensions with the universal role [11, 12] and/or inverse roles do. In fact, despite the fact that  $\mathcal{ALCO}$  and  $\mathcal{ALCHO}$  do not enjoy the BDP, we show that for all DLs considered in this paper the problem to decide the existence of an explicit definition of a concept name over the signature containing all symbols distinct from that concept name is  $\text{EXPTIME}$ -complete, thus not harder than deduction.

Detailed proofs are provided in a technical appendix submitted alongside this article.

## 2 Related Work

The CIP, PBDP, and BDP have been investigated extensively. In addition to the work discussed in the introduction, we mention the investigation of interpolation and definability in modal logic in general [41] and in hybrid modal logic in particular [1, 10]. Also related is work on interpolation in guarded logics [24, 23, 3, 5, 4].

Relevant work on Craig interpolation and Beth definability in description logic has been discussed in the introduction. Craig interpolation should not be confused with work on uniform interpolation, both in description logic [38, 40, 44, 33] and in modal logic [52, 34, 25]. Uniform interpolants generalize Craig interpolants in the sense that a uniform interpolant is an interpolant for a fixed antecedent and any formula implied by the antecedent and sharing with it a fixed set of symbols.

Interpolant and explicit definition existence have hardly been investigated for logics that do not enjoy the CIP or PBDP. Exceptions include linear temporal logic, LTL, for which the decidability of interpolant existence has been shown in [45, 21, 22] and the guarded fragment for which decidability and  $3\text{EXPTIME}$  completeness for interpolant existence are shown in [30]. This is in contrast to work on uniform interpolants in description logics which has in fact focused on the existence and computation of uniform interpolants that do not always exist.

Finally, we note that in [7, 8, 49], the authors propose the use of referring expressions in a query answering context with weaker DLs. The focus is on using functional roles to generate referring expressions for individuals for which there might not be a denoting individual name at all in the language.

### 3 Preliminaries

We use standard DL notation, see [2] for details. Let  $\mathbf{N}_C$ ,  $\mathbf{N}_R$ , and  $\mathbf{N}_I$  be mutually disjoint and countably infinite sets of *concept*, *role*, and *individual names*. A *role* is a role name  $s$  or an *inverse role*  $s^-$ , with  $s$  a role name and  $(s^-)^- = s$ . We use  $u$  to denote the *universal role*. A *nominal* takes the form  $\{a\}$ , with  $a$  an individual name. An  $\mathcal{ALC}\mathcal{I}\mathcal{O}^u$ -*concept* is defined according to the syntax rule

$$C, D ::= \top \mid A \mid \{a\} \mid \neg C \mid C \sqcap D \mid \exists r.C$$

where  $a$  ranges over individual names,  $A$  over concept names, and  $r$  over roles. We use  $C \sqcup D$  as abbreviation for  $\neg(\neg C \sqcap \neg D)$ ,  $C \rightarrow D$  for  $\neg C \sqcup D$ , and  $\forall r.C$  for  $\neg \exists r.(\neg C)$ . We use several fragments of  $\mathcal{ALC}\mathcal{I}\mathcal{O}^u$ , including  $\mathcal{ALC}\mathcal{I}\mathcal{O}$ , obtained by dropping the universal role,  $\mathcal{ALC}\mathcal{O}^u$ , obtained by dropping inverse roles,  $\mathcal{ALC}\mathcal{O}$ , obtained from  $\mathcal{ALC}\mathcal{O}^u$  by dropping the universal role, and  $\mathcal{ALC}$ , obtained from  $\mathcal{ALC}\mathcal{O}$  by dropping nominals. If  $\mathcal{L}$  is any of the DLs above, then an  $\mathcal{L}$ -*concept inclusion* ( $\mathcal{L}$ -*CI*) takes the form  $C \sqsubseteq D$  with  $C$  and  $D$   $\mathcal{L}$ -concepts. An  $\mathcal{L}$ -*ontology* is a finite set of  $\mathcal{L}$ -CIs. We also consider DLs with *role inclusions* (*RIs*), expressions of the form  $r \sqsubseteq s$ , where  $r$  and  $s$  are roles. As usual, the addition of RIs is indicated by adding the letter  $\mathcal{H}$  to the name of the DL, where inverse roles occur in RIs only if the DL admits inverse roles. Thus, for example,  $\mathcal{ALC}\mathcal{H}$ -*ontologies* are finite sets of  $\mathcal{ALC}$ -CIs and RIs not using inverse roles and  $\mathcal{ALC}\mathcal{H}\mathcal{I}\mathcal{O}^u$ -*ontologies* are finite sets of  $\mathcal{ALC}\mathcal{I}\mathcal{O}^u$ -CIs and RIs. In what follows we use  $\text{DL}_{\text{nr}}$  to denote the set of DLs  $\mathcal{ALC}\mathcal{O}$ ,  $\mathcal{ALC}\mathcal{I}\mathcal{O}$ ,  $\mathcal{ALC}\mathcal{H}$ ,  $\mathcal{ALC}\mathcal{H}\mathcal{O}$ ,  $\mathcal{ALC}\mathcal{H}\mathcal{I}\mathcal{O}$ , and their extensions with the universal role. To simplify notation we do not drop the letter  $\mathcal{H}$  when speaking about the concepts and CIs of a DL with RIs. Thus, for example, we sometimes use the expressions  $\mathcal{ALC}\mathcal{H}\mathcal{O}$ -concept and  $\mathcal{ALC}\mathcal{H}\mathcal{O}$ -CI to denote  $\mathcal{ALC}\mathcal{O}$ -concepts and CIs, respectively.

The semantics is defined in terms of *interpretations*  $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$  as usual, see [2]. An interpretation  $\mathcal{I}$  *satisfies* an  $\mathcal{L}$ -CI  $C \sqsubseteq D$  if  $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$  and an RI  $r \sqsubseteq s$  if  $r^{\mathcal{I}} \subseteq s^{\mathcal{I}}$ . We say that  $\mathcal{I}$  is a *model* of an ontology  $\mathcal{O}$  if it satisfies all inclusions in it. We say that an inclusion  $\alpha$  follows from an ontology  $\mathcal{O}$ , in symbols  $\mathcal{O} \models \alpha$ , if every model of  $\mathcal{O}$  satisfies  $\alpha$ . We write  $\mathcal{O} \models C \equiv D$  if  $\mathcal{O} \models C \sqsubseteq D$  and  $\mathcal{O} \models D \sqsubseteq C$ . A concept  $C$  is *satisfiable* w.r.t. an ontology  $\mathcal{O}$  if there is a model  $\mathcal{I}$  of  $\mathcal{O}$  with  $C^{\mathcal{I}} \neq \emptyset$ .

A *signature*  $\Sigma$  is a set of concept, role, and individual names, uniformly referred to as *symbols*. Following standard practice we do not regard the universal role as a symbol but as a logical connective. Thus, the universal role is not contained in any signature. We use  $\text{sig}(X)$  to denote the set of symbols used in any syntactic object  $X$  such as a concept or an ontology. An  $\mathcal{L}(\Sigma)$ -*concept* is an  $\mathcal{L}$ -concept  $C$  with  $\text{sig}(C) \subseteq \Sigma$ .

We next recall model-theoretic characterizations of when nodes in interpretations are indistinguishable by concepts formulated in one of the DLs  $\mathcal{L}$  introduced above. A *pointed interpretation* is a pair  $\mathcal{I}, d$  with  $\mathcal{I}$  an interpretation and  $d \in \Delta^{\mathcal{I}}$ . For pointed interpretations  $\mathcal{I}, d$  and  $\mathcal{J}, e$  and a signature  $\Sigma$ , we write  $\mathcal{I}, d \equiv_{\mathcal{L}, \Sigma} \mathcal{J}, e$  and say that  $\mathcal{I}, d$  and  $\mathcal{J}, e$  are  $\mathcal{L}(\Sigma)$ -*equivalent* if  $d \in C^{\mathcal{I}}$  iff  $e \in C^{\mathcal{J}}$ , for all  $\mathcal{L}(\Sigma)$ -concepts  $C$ .

[AtomC]	for all $(d, e) \in S$ : $d \in A^{\mathcal{I}}$ iff $e \in A^{\mathcal{J}}$
[AtomI]	for all $(d, e) \in S$ : $d = a^{\mathcal{I}}$ iff $e = a^{\mathcal{J}}$
[Forth]	if $(d, e) \in S$ and $(d, d') \in r^{\mathcal{I}}$ , then there is a $e'$ with $(e, e') \in r^{\mathcal{J}}$ and $(d', e') \in S$ .
[Back]	if $(d, e) \in S$ and $(e, e') \in r^{\mathcal{J}}$ , then there is a $d'$ with $(d, d') \in r^{\mathcal{I}}$ and $(d', e') \in S$ .

**Fig. 1.** Conditions on  $S \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{J}}$ .

As for the model-theoretic characterizations, we start with  $\mathcal{ALC}$ . Let  $\Sigma$  be a signature. A relation  $S \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{J}}$  is an  $\mathcal{ALC}(\Sigma)$ -bisimulation if conditions [AtomC], [Forth] and [Back] from Figure 1 hold, where  $A$  and  $r$  range over all concept and role names in  $\Sigma$ , respectively. We write  $\mathcal{I}, d \sim_{\mathcal{ALC}, \Sigma} \mathcal{J}, e$  and call  $\mathcal{I}, d$  and  $\mathcal{J}, e$   $\mathcal{ALC}(\Sigma)$ -bisimilar if there exists an  $\mathcal{ALC}(\Sigma)$ -bisimulation  $S$  such that  $(d, e) \in S$ . For  $\mathcal{ALCO}$ , we define  $\sim_{\mathcal{ALCO}, \Sigma}$  analogously, but now demand that, in Figure 1, also condition [AtomI] holds for all individual names  $a \in \Sigma$ . For languages  $\mathcal{L}$  with inverse roles, we demand that, in Figure 1,  $r$  additionally ranges over inverse roles. For languages  $\mathcal{L}$  with the universal role we extend the respective conditions by demanding that the domain  $\text{dom}(S)$  and range  $\text{ran}(S)$  of  $S$  contain  $\Delta^{\mathcal{I}}$  and  $\Delta^{\mathcal{J}}$ , respectively. If a DL  $\mathcal{L}$  has RIs, then we use  $\mathcal{I}, d \sim_{\mathcal{L}, \Sigma} \mathcal{J}, e$  to state that  $\mathcal{I}, d \sim_{\mathcal{L}', \Sigma} \mathcal{J}, e$  for the fragment  $\mathcal{L}'$  of  $\mathcal{L}$  without RIs.

The next lemma summarizes the model-theoretic characterizations for all relevant DLs [37, 20]. For the definition of  $\omega$ -saturated structures, we refer the reader to [15].

**Lemma 1.** *Let  $\mathcal{I}, d$  and  $\mathcal{J}, e$  be pointed interpretations and  $\omega$ -saturated. Let  $\mathcal{L} \in \text{DL}_{\text{nr}}$  and  $\Sigma$  a signature. Then*

$$\mathcal{I}, d \equiv_{\mathcal{L}, \Sigma} \mathcal{J}, e \quad \text{iff} \quad \mathcal{I}, d \sim_{\mathcal{L}, \Sigma} \mathcal{J}, e.$$

*For the “if”-direction, the  $\omega$ -saturatedness condition can be dropped.*

## 4 Craig Interpolation and Beth Definability

We introduce the Craig interpolation property (CIP) and the (projective) Beth definability property ((P)BDP) as defined in [13]. We recall their relationship and observe that no DL in  $\text{DL}_{\text{nr}}$  enjoys the CIP or PBDP while all except  $\mathcal{ALCO}$  and  $\mathcal{ALCHO}$  enjoy the BDP.

Let  $\mathcal{O}_1, \mathcal{O}_2$  be  $\mathcal{L}$ -ontologies and let  $C_1, C_2$  be  $\mathcal{L}$ -concepts. We set  $\text{sig}(\mathcal{O}, C) = \text{sig}(\mathcal{O}) \cup \text{sig}(C)$ , for any ontology  $\mathcal{O}$  and concept  $C$ . Then an  $\mathcal{L}$ -concept  $D$  is called an  $\mathcal{L}$ -interpolant for  $C_1 \sqsubseteq C_2$  under  $\mathcal{O}_1 \cup \mathcal{O}_2$  if

- $\text{sig}(D) \subseteq \text{sig}(\mathcal{O}_1, C_1) \cap \text{sig}(\mathcal{O}_2, C_2)$ ;
- $\mathcal{O}_1 \cup \mathcal{O}_2 \models C_1 \sqsubseteq D$ ;
- $\mathcal{O}_1 \cup \mathcal{O}_2 \models D \sqsubseteq C_2$ .

**Definition 1.** A DL  $\mathcal{L}$  has the Craig interpolation property (CIP) if for any  $\mathcal{L}$ -ontologies  $\mathcal{O}_1, \mathcal{O}_2$  and  $\mathcal{L}$ -concepts  $C_1, C_2$  such that  $\mathcal{O}_1 \cup \mathcal{O}_2 \models C_1 \sqsubseteq C_2$  there exists an  $\mathcal{L}$ -interpolant for  $C_1 \sqsubseteq C_2$  under  $\mathcal{O}_1 \cup \mathcal{O}_2$ .

We next define the relevant definability notions. Let  $\mathcal{O}$  be an ontology and  $C$  a concept. Let  $\Sigma \subseteq \text{sig}(\mathcal{O}, C)$  be a signature. An  $\mathcal{L}(\Sigma)$ -concept  $D$  is an *explicit  $\mathcal{L}(\Sigma)$ -definition of  $C$  under  $\mathcal{O}$*  if  $\mathcal{O} \models C \equiv D$ . We call  $C$  *explicitly definable in  $\mathcal{L}(\Sigma)$  under  $\mathcal{O}$*  if there is an explicit  $\mathcal{L}(\Sigma)$ -definition of  $C$  under  $\mathcal{O}$ . The  $\Sigma$ -reduct  $\mathcal{I}_\Sigma$  of an interpretation  $\mathcal{I}$  coincides with  $\mathcal{I}$  except that no non- $\Sigma$  symbol is interpreted in  $\mathcal{I}_\Sigma$ . A concept  $C$  is called *implicitly definable from  $\Sigma$  under  $\mathcal{O}$*  if the  $\Sigma$ -reduct of any model  $\mathcal{I}$  of  $\mathcal{O}$  determines the set  $C^\mathcal{I}$ ; in other words, if  $\mathcal{I}$  and  $\mathcal{J}$  are both models of  $\mathcal{O}$  such that  $\mathcal{I}_\Sigma = \mathcal{J}_\Sigma$ , then  $C^\mathcal{I} = C^\mathcal{J}$ . It is easy to see that implicit definability can be reformulated as a standard reasoning problem as follows: a concept  $C$  is implicitly definable from  $\Sigma$  under  $\mathcal{O}$  iff  $\mathcal{O} \cup \mathcal{O}_\Sigma \models C \equiv C_\Sigma$ , where  $\mathcal{O}_\Sigma$  and  $C_\Sigma$  are obtained from  $\mathcal{O}$  and, respectively,  $C$  by replacing every non- $\Sigma$  symbol uniformly by a fresh symbol. If a concept is explicitly definable in  $\mathcal{L}(\Sigma)$  under  $\mathcal{O}$ , then it is implicitly definable from  $\Sigma$  under  $\mathcal{O}$ , for any language  $\mathcal{L}$ . A logic enjoys the projective Beth definability property if the converse implication holds as well:

**Definition 2.** A DL  $\mathcal{L}$  has the projective Beth definability property (PBDP) if for any  $\mathcal{L}$ -ontology  $\mathcal{O}$ ,  $\mathcal{L}$ -concept  $C$ , and signature  $\Sigma \subseteq \text{sig}(\mathcal{O}, C)$  the following holds: if  $C$  is implicitly definable from  $\Sigma$  under  $\mathcal{O}$ , then  $C$  is explicitly  $\mathcal{L}(\Sigma)$ -definable under  $\mathcal{O}$ .

It is known that the CIP and PBDP are tightly linked [13].

**Lemma 2.** *If  $\mathcal{L}$  enjoys the CIP, then  $\mathcal{L}$  enjoys the PBDP.*

To see this, assume that an  $\mathcal{L}$ -concept  $C$  is implicitly definable from  $\Sigma$  under an  $\mathcal{L}$ -ontology  $\mathcal{O}$ , for some signature  $\Sigma$ . Then  $\mathcal{O} \cup \mathcal{O}_\Sigma \models C \equiv C_\Sigma$ , with  $\mathcal{O}_\Sigma$  and  $C_\Sigma$  as above. Take an  $\mathcal{L}$ -interpolant  $D$  for  $C \sqsubseteq C_\Sigma$  under  $\mathcal{O} \cup \mathcal{O}_\Sigma$ . Then  $D$  is an explicit  $\mathcal{L}(\Sigma)$ -definition of  $C$  under  $\mathcal{O}$ .

An important special case of explicit definability is the explicit definability of a concept name  $A$  from  $\text{sig}(\mathcal{O}) \setminus \{A\}$  under an ontology  $\mathcal{O}$ . For this case, we consider the following *non-projective version* of the Beth definability property: A DL  $\mathcal{L}$  enjoys the Beth definability property (BDP) if for any  $\mathcal{L}$ -ontology  $\mathcal{O}$  and any concept name  $A$  the following holds: if  $A$  is implicitly definable from  $\text{sig}(\mathcal{O}) \setminus \{A\}$  under  $\mathcal{O}$ , then  $A$  is explicitly  $\mathcal{L}(\text{sig}(\mathcal{O}) \setminus \{A\})$ -definable under  $\mathcal{O}$ . Clearly, the PBDP implies the BDP, but not vice versa.

Many DLs including  $\mathcal{ALC}$ ,  $\mathcal{ALCI}$ , and  $\mathcal{ALCI}^u$ , enjoy CIP and (P)BDP. However, DLs supporting nominals or role inclusions do not enjoy PBDP and thus, by Lemma 2, also not the CIP [13]. The following theorem summarizes the situation.

**Theorem 3.** (1) *No  $\mathcal{L} \in \text{DL}_{\text{nr}}$  enjoys the CIP or the PBDP.*  
(2) *All  $\mathcal{L} \in \text{DL}_{\text{nr}} \setminus \{\mathcal{ALCO}, \mathcal{ALCHO}\}$  enjoy the BDP.  $\mathcal{ALCO}$  and  $\mathcal{ALCHO}$  do not enjoy the BDP.*

To see Part (1) of the theorem, we provide two example ontologies. Consider first the  $\mathcal{ALCO}$ -ontology

$$\mathcal{O}_1 = \{ \{a\} \sqsubseteq \exists r.\{a\}, A \sqcap \neg\{a\} \sqsubseteq \forall r.(\neg\{a\} \rightarrow \neg A), \\ \neg A \sqcap \neg\{a\} \sqsubseteq \forall r.(\neg\{a\} \rightarrow A) \}.$$

Thus,  $\mathcal{O}_1$  implies that  $a$  is reflexive and that no node distinct from  $a$  is reflexive. Let  $\Sigma = \{r, A\}$ . Then  $\{a\}$  is implicitly definable from  $\Sigma$  under  $\mathcal{O}_1$  since  $\mathcal{O}_1 \models \forall x((x = a) \leftrightarrow r(x, x))$ , but one can show that  $\{a\}$  is not explicitly  $\mathcal{L}(\Sigma)$ -definable under  $\mathcal{O}_1$  for any  $\mathcal{L} \in \text{DL}_{nr}$  with nominals.

Consider now the  $\mathcal{ALCH}$ -ontology

$$\mathcal{O}_2 = \{ r \sqsubseteq r_1, r \sqsubseteq r_2, \neg\exists r.\top \sqcap \exists r_1.A \sqsubseteq \forall r_2.\neg A, \\ \neg\exists r.\top \sqcap \exists r_1.\neg A \sqsubseteq \forall r_2.A \}$$

from [13]. Then  $\exists r.\top$  is implicitly definable from  $\Sigma = \{r_1, r_2\}$  under  $\mathcal{O}_2$  as  $\mathcal{O}_2 \models \forall x(\exists y r(x, y) \leftrightarrow \exists y (r_1(x, y) \wedge r_2(x, y)))$ , but it is not explicitly definable from  $\Sigma$  under  $\mathcal{O}_2$  in any DL from  $\text{DL}_{nr}$ .

Part (2) of Theorem 3 for  $\mathcal{L} \in \text{DL}_{nr}$  without nominals or with the universal role follows from Theorems 2.5.4 and 6.2.4 in [11], respectively, see also [12]. That  $\mathcal{ALCO}$  and  $\mathcal{ALCHO}$  do not enjoy the BDP is shown in [12]. It remains to prove that  $\mathcal{ALCIO}$  and  $\mathcal{ALCHIO}$  enjoy the BDP. This is done in the appendix using a generalized version of cartesian products called bisimulation products.

## 5 Notions Studied and Main Result

The failure of CIP and (P)BDP reported in Theorem 3 motivates the investigation of the respective decision problems of *interpolant existence* and *projective and non-projective definition existence*, which are defined as follows.

**Definition 3.** Let  $\mathcal{L}$  be a DL. Then  $\mathcal{L}$ -*interpolant existence* is the problem to decide for any  $\mathcal{L}$ -ontologies  $\mathcal{O}_1, \mathcal{O}_2$  and  $\mathcal{L}$ -concepts  $C_1, C_2$  whether there exists an  $\mathcal{L}$ -interpolant for  $C_1 \sqsubseteq C_2$  under  $\mathcal{O}_1 \cup \mathcal{O}_2$ .

**Definition 4.** Let  $\mathcal{L}$  be a DL. *Projective  $\mathcal{L}$ -definition existence* is the problem to decide for any  $\mathcal{L}$ -ontology  $\mathcal{O}$ ,  $\mathcal{L}$ -concept  $C$ , and signature  $\Sigma \sqsubseteq \text{sig}(\mathcal{O}, C)$  whether there exists an explicit  $\mathcal{L}(\Sigma)$ -definition of  $C$  under  $\mathcal{O}$ . (*Non-projective  $\mathcal{L}$ -definition existence* is the sub-problem where  $C$  ranges only over concept names  $A$  and  $\Sigma = \text{sig}(\mathcal{O}) \setminus \{A\}$ ).

Observe that interpolant existence reduces to checking  $\mathcal{O}_1 \cup \mathcal{O}_2 \models C_1 \sqsubseteq C_2$  for logics with the CIP but that this is not the case for logics without the CIP. Similarly, projective definition existence reduces to checking implicit definability for logics with the PBDP but not for logics without the PBDP. Also observe that the following reduction can be proved similarly to the proof of Lemma 2.

**Lemma 4.** *Let  $\mathcal{L}$  be a DL. There is a polynomial time reduction of projective  $\mathcal{L}$ -definition existence to  $\mathcal{L}$ -interpolant existence.*

In terms of applications of the introduced decision problems, we note that non-projective definition existence is particularly relevant for the extraction of acyclic terminologies from ontologies [12], while the flexibility of projective definition existence is useful in most other applications discussed in the introduction. When it comes to computing referring expressions as discussed in the introduction, we are interested in the case when  $C$  ranges over nominals  $\{a\}$ . We then speak of *projective  $\mathcal{L}$ -referring expression existence* and of *(non-projective)  $\mathcal{L}$ -referring expression existence*, if  $\Sigma = \text{sig}(\mathcal{O}) \setminus \{a\}$ .

The main concern of the present paper is to study the computational complexity of the introduced decision problems. As a preliminary step, we provide model-theoretic characterizations for the existence of interpolants and explicit definitions in terms of bisimulations as captured in the following central notion.

**Definition 5 (Joint consistency).** Let  $\mathcal{L} \in \text{DL}_{\text{nr}}$ . Let  $\mathcal{O}_1, \mathcal{O}_2$  be  $\mathcal{L}$ -ontologies,  $C_1, C_2$  be  $\mathcal{L}$ -concepts, and  $\Sigma \subseteq \text{sig}(\mathcal{O}_1, \mathcal{O}_2, C_1, C_2)$  be a signature. Then  $\mathcal{O}_1, C_1$  and  $\mathcal{O}_2, C_2$  are called *jointly consistent modulo  $\mathcal{L}(\Sigma)$ -bisimulations* if there exist pointed models  $\mathcal{I}_1, d_1$  and  $\mathcal{I}_2, d_2$  such that  $\mathcal{I}_i$  is a model of  $\mathcal{O}_i$ ,  $d_i \in C_i^{\mathcal{I}_i}$ , for  $i = 1, 2$ , and  $\mathcal{I}_1, d_1 \sim_{\mathcal{L}, \Sigma} \mathcal{I}_2, d_2$ .

The associated decision problem, *joint consistency modulo  $\mathcal{L}$ -bisimulations*, is defined in the expected way. The following result characterizes the existence of interpolants using joint consistency modulo  $\mathcal{L}(\Sigma)$ -bisimulations. The proof uses Lemma 1.

**Theorem 5.** Let  $\mathcal{L} \in \text{DL}_{\text{nr}}$ . Let  $\mathcal{O}_1, \mathcal{O}_2$  be  $\mathcal{L}$ -ontologies and let  $C_1, C_2$  be  $\mathcal{L}$ -concepts, and  $\Sigma = \text{sig}(\mathcal{O}_1, C_1) \cap \text{sig}(\mathcal{O}_2, C_2)$ . Then the following conditions are equivalent:

1. there is no  $\mathcal{L}$ -interpolant for  $C_1 \sqsubseteq C_2$  under  $\mathcal{O}_1 \cup \mathcal{O}_2$ ;
2.  $\mathcal{O}_1 \cup \mathcal{O}_2, C_1$  and  $\mathcal{O}_1 \cup \mathcal{O}_2, \neg C_2$  are jointly consistent modulo  $\mathcal{L}(\Sigma)$ -bisimulations.

The following characterization of the existence of explicit definitions is a direct consequence of Theorem 5.

**Theorem 6.** Let  $\mathcal{L} \in \text{DL}_{\text{nr}}$ . Let  $\mathcal{O}$  be an  $\mathcal{L}$ -ontology,  $C$  an  $\mathcal{L}$ -concept, and  $\Sigma \subseteq \text{sig}(\mathcal{O}, C)$  a signature. Then the following conditions are equivalent:

1. there is no explicit  $\mathcal{L}(\Sigma)$ -definition of  $C$  under  $\mathcal{O}$ ;
2.  $\mathcal{O}, C$  and  $\mathcal{O}, \neg C$  are jointly consistent modulo  $\mathcal{L}(\Sigma)$ -bisimulations.

In the remainder of the paper we establish the following tight complexity results for the introduced decision problems.

**Theorem 7.** Let  $\mathcal{L} \in \text{DL}_{\text{nr}}$ . Then, (i)  $\mathcal{L}$ -interpolant existence and projective  $\mathcal{L}$ -definition existence are 2EXPTIME-complete, (ii) if  $\mathcal{L}$  admits nominals, then both projective and non-projective  $\mathcal{L}$ -referring expression existence are 2EXPTIME-complete, and (iii) non-projective  $\mathcal{L}$ -definition existence is EXPTIME-complete.

Observe that the characterizations given in Theorems 5 and 6 provide reductions of interpolant and definition existence to the complement of joint consistency modulo  $\mathcal{L}$ -bisimulations. Hence for the upper bounds in Points (i) and (ii), it suffices to decide the latter in double exponential time which is what we do in the next section. After that, we provide lower bounds for definition existence and referring expression existence which imply the corresponding lower bounds for interpolant existence via Lemma 4. Finally, we show the upper bounds of Point (iii); the lower bounds are inherited from validity.

## 6 The 2EXPTIME upper bound

We provide a double exponential time mosaic-style algorithm that decides joint consistency modulo  $\mathcal{L}$ -bisimulations, for all  $\mathcal{L} \in \text{DL}_{\text{nr}}$ .

**Theorem 8.** *Let  $\mathcal{L} \in \text{DL}_{\text{nr}}$ . Then joint consistency modulo  $\mathcal{L}$ -bisimulations is in 2EXPTIME.*

Assume  $\mathcal{L} \in \text{DL}_{\text{nr}}$ . We may assume that  $\mathcal{L}$  extends  $\mathcal{ALCHO}$ . Consider  $\mathcal{L}$ -ontologies  $\mathcal{O}_1$  and  $\mathcal{O}_2$  and  $\mathcal{L}$ -concepts  $C_1$  and  $C_2$ . Let  $\Sigma \subseteq \text{sig}(\mathcal{O}_1, \mathcal{O}_2, C_1, C_2)$  be a signature. Let  $\Xi = \text{sub}(\mathcal{O}_1, \mathcal{O}_2, C_1, C_2)$  denote the closure under single negation of the set of subconcepts of concepts in  $\mathcal{O}_1, \mathcal{O}_2, C_1, C_2$ . A  $\Xi$ -type  $t$  is a subset of  $\Xi$  such that there exists an interpretation  $\mathcal{I}$  and  $d \in \Delta^{\mathcal{I}}$  with  $t = \text{tp}_{\Xi}(\mathcal{I}, d)$ , where

$$\text{tp}_{\Xi}(\mathcal{I}, d) = \{C \in \Xi \mid d \in C^{\mathcal{I}}\}$$

is the  $\Xi$ -type realized at  $d$  in  $\mathcal{I}$ . Let  $T(\Xi)$  denote the set of all  $\Xi$ -types. Let  $r$  be a role. A pair  $(t_1, t_2)$  of  $\Xi$ -types  $t_1, t_2$  is  $r$ -coherent for  $\mathcal{O}_i$ , in symbols  $t_1 \rightsquigarrow_{r, \mathcal{O}_i} t_2$ , if the following condition holds for all roles  $s$  with  $\mathcal{O}_i \models r \sqsubseteq s$ : (1) if  $\neg \exists s.C \in t_1$ , then  $C \notin t_2$  and (2) if  $\neg \exists s^-.C \in t_2$ , then  $C \notin t_1$ . We aim to work with pairs  $(T_1, T_2) \in 2^{T(\Xi)} \times 2^{T(\Xi)}$  such that all  $t \in T_1 \cup T_2$  are realized in mutually  $\mathcal{L}(\Sigma)$ -bisimilar nodes of models of  $\mathcal{O}_i$ , for  $i = 1, 2$ .

Thus, we now formulate conditions on a set  $\mathcal{S} \subseteq 2^{T(\Xi)} \times 2^{T(\Xi)}$  which ensure that one can construct from  $\mathcal{S}$  models  $\mathcal{I}_i$  of  $\mathcal{O}_i$  such that for any pair  $(T_1, T_2) \in \mathcal{S}$  and all  $t \in T_i$ ,  $i = 1, 2$ , there are nodes  $d_t \in \Delta^{\mathcal{I}_i}$  realizing  $t$ , such that all  $d_t$ ,  $t \in T_1 \cup T_2$  are mutually  $\mathcal{L}(\Sigma)$ -bisimilar. We lift the definition of  $r$ -coherence from pairs of types to pairs of elements of  $2^{T(\Xi)} \times 2^{T(\Xi)}$ . Let  $r$  be a role. We call a pair  $(T_1, T_2), (T'_1, T'_2)$   $r$ -coherent, in symbols  $(T_1, T_2) \rightsquigarrow_r (T'_1, T'_2)$ , if for  $i = 1, 2$  and any  $t \in T_i$  there exists a  $t' \in T'_i$  such that  $t \rightsquigarrow_{r, \mathcal{O}_i} t'$ . Moreover, to deal with DLs with inverse roles, we say that  $(T_1, T_2), (T'_1, T'_2)$  are *fully  $r$ -coherent*, in symbols  $(T_1, T_2) \leftrightarrow_r (T'_1, T'_2)$  if the converse holds as well: for  $i = 1, 2$  and any  $t' \in T'_i$  there exists a  $t \in T_i$  such that  $t \rightsquigarrow_{r, \mathcal{O}_i} t'$ .

We first formulate conditions that ensure that nominals are interpreted as singletons and that individuals in  $\Sigma$  are preserved by the bisimulation. Say that  $\mathcal{S}$  is *good for nominals* if for every individual name  $a \in \text{sig}(\Xi)$  and  $i = 1, 2$  there exists exactly one  $t_a^i$  with  $\{a\} \in t_a^i \in \bigcup_{(T_1, T_2) \in \mathcal{S}} T_i$  and exactly one pair  $(T_1, T_2) \in \mathcal{S}$  with  $t_a^i \in T_i$ . Moreover, if  $a \in \Sigma$ , then that pair either takes the form  $(\{t_a^1\}, \{t_a^2\})$  or the form  $(\{t_a^1\}, \emptyset)$  and  $(\emptyset, \{t_a^2\})$ , respectively.

Secondly, we ensure that the types used in  $\mathcal{S}$  are consistent with  $\mathcal{O}_1$  and  $\mathcal{O}_2$ , respectively. Say that  $\mathcal{S}$  is *good for  $\mathcal{O}_1, \mathcal{O}_2$*  if  $(\emptyset, \emptyset) \notin \mathcal{S}$  and for every  $(T_1, T_2) \in \mathcal{S}$  all types  $t \in T_i$  are realizable in a model of  $\mathcal{O}_i$ ,  $i = 1, 2$ .

Finally, we need to ensure that concept names in  $\Sigma$  are preserved by the bisimulation and that the back and forth condition of bisimulations hold.  $\mathcal{S}$  is called  *$\mathcal{ALCHO}(\Sigma)$ -good* if it is good for nominals and  $\mathcal{O}_1, \mathcal{O}_2$ , and the following conditions hold:

1.  *$\Sigma$ -concept name coherence*: for any concept name  $A \in \Sigma$  and  $(T_1, T_2) \in \mathcal{S}$ ,  $A \in t$  iff  $A \in t'$  for all  $t, t' \in T_1 \cup T_2$ ;
2. *Existential saturation*: for  $i = 1, 2$ , if  $(T_1, T_2) \in \mathcal{S}$  and  $\exists r.C \in t \in T_i$ , then there exists  $(T'_1, T'_2) \in \mathcal{S}$  such that (1) there exists  $t' \in T'_i$  with  $C \in t'$  and  $t \rightsquigarrow_{r, \mathcal{O}_i} t'$  and (2) if  $\mathcal{O}_i \models r \sqsubseteq s$  with  $s \in \Sigma$ , then  $(T_1, T_2) \rightsquigarrow_s (T'_1, T'_2)$ .

If inverse roles or the universal role are present then we strengthen Condition 2 to Condition 2 $\mathcal{I}$  and add Condition 3 $u$ , respectively:

- 2 $\mathcal{I}$ . Condition 2 with ' $s \in \Sigma$ ' replaced by ' $s$  a role over  $\Sigma$ ' and ' $(T_1, T_2) \rightsquigarrow_s (T'_1, T'_2)$ ' replaced by ' $(T_1, T_2) \rightsquigarrow_s (T'_1, T'_2)$ '.
- 3 $u$ . if  $(T_1, T_2) \in \mathcal{S}$ , then  $T_i \neq \emptyset$ , for  $i = 1, 2$ .

Thus,  $\mathcal{S}$  is  *$\mathcal{ALCHIO}(\Sigma)$ -good* if the conditions above hold with Condition 2 replaced by Condition 2 $\mathcal{I}$  and  $\mathcal{S}$  is  *$\mathcal{ALCHO}^u(\Sigma)$ -good* and, respectively,  *$\mathcal{ALCHIO}^u(\Sigma)$ -good* if also Condition 3 $u$  holds.

**Lemma 9.** *Let  $\mathcal{L} \in \text{DL}_{\text{nr}}$ . Assume  $\mathcal{O}_1, \mathcal{O}_2$  are  $\mathcal{L}$ -ontologies,  $C_1, C_2$  are  $\mathcal{L}$ -concepts, and let  $\Sigma \subseteq \text{sig}(\Xi)$  be a signature. The following conditions are equivalent:*

1.  $\mathcal{O}_1, C_1$  and  $\mathcal{O}_2, C_2$  are jointly consistent modulo  $\mathcal{L}(\Sigma)$ -bisimulations.
2. there exists an  $\mathcal{L}(\Sigma)$ -good set  $\mathcal{S}$  and  $\Xi$ -types  $t_1, t_2$  with  $C_1 \in t_1$  and  $C_2 \in t_2$  such that  $t_1 \in T_1$  and  $t_2 \in T_2$  for some  $(T_1, T_2) \in \mathcal{S}$ .

**Proof.** (sketch) " $1 \Rightarrow 2$ ". Let  $\mathcal{I}_1, d_1 \sim_{\mathcal{L}, \Sigma} \mathcal{I}_2, d_2$  for models  $\mathcal{I}_1$  of  $\mathcal{O}_1$  and  $\mathcal{I}_2$  of  $\mathcal{O}_2$  such that  $d_1, d_2$  realize  $\Xi$ -types  $t_1, t_2$  and  $C_1 \in t_1, C_2 \in t_2$ . Define  $\mathcal{S}$  by setting  $(T_1, T_2) \in \mathcal{S}$  if there is  $d \in \Delta^{\mathcal{I}_i}$  for some  $i \in \{1, 2\}$  such that

$$T_j = \{\text{tp}_{\Xi}(\mathcal{I}_j, e) \mid e \in \Delta^{\mathcal{I}_j}, \mathcal{I}_i, d \sim_{\mathcal{L}, \Sigma} \mathcal{I}_j, e\},$$

for  $j = 1, 2$ . One can show that  $\mathcal{S}$  is  $\mathcal{L}(\Sigma)$ -good and satisfies Condition 2.

" $2 \Rightarrow 1$ ". Assume  $\mathcal{S}$  is  $\mathcal{L}(\Sigma)$ -good and we have  $\Xi$ -types  $s_1, s_2$  with  $C_1 \in s_1$  and  $C_2 \in s_2$  such that  $s_1 \in S_1$  and  $s_2 \in S_2$  for some  $(S_1, S_2) \in \mathcal{S}$ . If  $\mathcal{L}$  does not admit inverse roles, then interpretations  $\mathcal{I}_1$  and  $\mathcal{I}_2$  witnessing Condition 1 are defined by setting

$$\begin{aligned} \Delta^{\mathcal{I}_i} &:= \{(t, (T_1, T_2)) \mid t \in T_i \text{ and } (T_1, T_2) \in \mathcal{S}\} \\ r^{\mathcal{I}_i} &:= \{((t, p), (t', p')) \in \Delta^{\mathcal{I}_i} \times \Delta^{\mathcal{I}_i} \mid t \rightsquigarrow_{r, \mathcal{O}_i} t', \\ &\quad \forall s \in \Sigma ((\mathcal{O}_i \models r \sqsubseteq s) \Rightarrow p \rightsquigarrow_s p')\} \\ A^{\mathcal{I}_i} &:= \{(t, p) \in \Delta^{\mathcal{I}_i} \mid A \in t\} \\ a^{\mathcal{I}_i} &:= (t, (T_1, T_2)) \in \Delta^{\mathcal{I}_i}, \{a\} \in t \in T_i \end{aligned}$$

If  $\mathcal{L}$  admits inverse roles then replace ‘ $s \in \Sigma$ ’ by ‘ $s$  a role over  $\Sigma$ ’ and ‘ $p \rightsquigarrow_s p'$ ’ by ‘ $p \leftrightarrow_s p'$ ’ in the definition of  $r^{\mathcal{I}_i}$ .  $\square$

The following lemma can now be established using a standard recursive bad mosaic elimination procedure.

**Lemma 10.** *Let  $\mathcal{L} \in \text{DL}_{\text{nr}}$ . Then it is decidable in double exponential time whether for  $\mathcal{L}$ -ontologies  $\mathcal{O}_1, \mathcal{O}_2$ ,  $\mathcal{L}$ -concepts  $C_1, C_2$ , and a signature  $\Sigma \subseteq \text{sig}(\Xi)$  there exists an  $\mathcal{S}$  and  $t_1, t_2$  satisfying Condition 2 of Lemma 9.*

Theorem 8 is a direct consequence of Lemmas 9 and 10.

## 7 The 2EXPTIME lower bound

We show that for any  $\mathcal{L}$  in  $\text{DL}_{\text{nr}}$ , projective  $\mathcal{L}$ -definition existence is 2EXPTIME-hard and that, if  $\mathcal{L}$  supports nominals, even (non-projective)  $\mathcal{L}$ -referring expression existence is 2EXPTIME-hard.

### 7.1 DLs with Nominals

We start with DLs with nominals. By Theorems 5 and 6, it suffices to prove the following result.

**Lemma 11.** *Let  $\mathcal{L} \in \text{DL}_{\text{nr}}$  admit nominals. It is 2EXPTIME-hard to decide for an  $\mathcal{L}$ -ontology  $\mathcal{O}$ , individual name  $b$ , and signature  $\Sigma \subseteq \text{sig}(\mathcal{O}) \setminus \{b\}$  whether  $\mathcal{O}, \{b\}$  and  $\mathcal{O}, \neg\{b\}$  are jointly consistent modulo  $\mathcal{L}(\Sigma)$ -bisimulations. This is true even if  $b$  is the only individual in  $\mathcal{O}$  and  $\Sigma = \text{sig}(\mathcal{O}) \setminus \{b\}$ .*

We reduce the word problem for  $2^n$ -space bounded alternating Turing machines, which is known to be 2EXPTIME-hard [14]. An *alternating Turing machine (ATM)* is a tuple  $M = (Q, \Theta, \Gamma, q_0, \Delta)$  where  $Q$  is the set of states consisting of *existential* and *universal states*,  $\Theta$  and  $\Gamma$  are input and tape alphabet,  $q_0 \in Q$  is the initial state, and the transition relation  $\Delta$  makes sure that existential and universal states alternate. We assume binary branching. The acceptance condition of our ATMs is defined in a slightly unusual way, without using accepting states: The ATM accepts if it runs forever on all branches and rejects otherwise. This is without loss of generality, since starting from the standard ATM model, this can be achieved by assuming that the ATM terminates on every input and then modifying it to enter an infinite loop from the accepting state. For a precise definition of ATMs and their acceptance condition, we refer the reader to the appendix.

The idea of the reduction is as follows. Given input word  $w$  of length  $n$ , we construct an ontology  $\mathcal{O}$  such that  $M$  accepts  $w$  iff  $\mathcal{O}, \{b\}$  and  $\mathcal{O}, \neg\{b\}$  are jointly consistent modulo  $\mathcal{L}(\Sigma)$ -bisimulations, where

$$\Sigma = \{r, s, Z, B_{\forall}, B_{\exists}^1, B_{\exists}^2\} \cup \{A_{\sigma} \mid \sigma \in \Gamma \cup (Q \times \Gamma)\}.$$

We provide the reduction here for  $\mathcal{L} = \mathcal{ALCCO}$ ; the modifications required for  $\Sigma = \text{sig}(\mathcal{O}) \setminus \{b\}$ , inverse roles, and the universal role are given in the appendix.

The ontology  $\mathcal{O}$  enforces that  $r(b, b)$  holds using the CI  $\{b\} \sqsubseteq \exists r.\{b\}$ . Moreover, any node distinct from  $b$  with an  $r$ -successor lies on an infinite  $r$ -path  $\rho$ , enforced by the CIs:

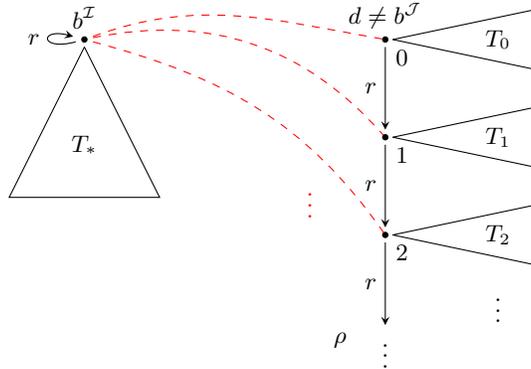
$$\neg\{b\} \sqcap \exists r.\top \sqsubseteq I_s \qquad I_s \sqsubseteq \exists r.\top \sqcap \forall r.I_s$$

Thus, if there exist models  $\mathcal{I}$  and  $\mathcal{J}$  of  $\mathcal{O}$  such that  $\mathcal{I}, b^{\mathcal{I}} \sim_{\mathcal{ALCO}(\Sigma)} \mathcal{J}, d$  for some  $d \neq b^{\mathcal{J}}$  it follows that in  $\mathcal{J}$  all nodes on some  $r$ -path  $\rho$  through  $d$  are  $\mathcal{ALCO}(\Sigma)$ -bisimilar. The situation is illustrated in Figure 2 where dashed edges represent the enforced bisimulation. In each point of  $\rho$  starts an infinite tree along role  $s$  that is supposed to mimick the computation of  $M$ : a configuration of  $M$  is represented by  $2^n$  consecutive elements of this infinite tree and is encoded by the concept names  $A_\sigma \in \Sigma$ . Moreover, each configuration is labeled by  $B_\forall$  (if it is universal) and  $B_\exists^i$  (if it is existential;  $i \in \{1, 2\}$  refers to the existential choice that is taken). All these trees, called  $T_*$  and  $T_i$  in Figure 2, have identical  $\Sigma$ -decorations due to the enforced bisimulation.

To coordinate successor configurations, we proceed as follows. Along  $\rho$ , a counter counts modulo  $2^n$  using concept names not in  $\Sigma$ . Along the trees  $T_i$ , two counters are maintained:

- one counter starting at 0 and counting modulo  $2^n$  to divide the tree in configurations of length  $2^n$ ;
- another counter starting at *the value of the counter on  $\rho$*  and also counting modulo  $2^n$ .

As on the  $i$ th  $s$ -tree  $T_i$  the second counter starts at all nodes at distances  $k \times 2^n - i$ , for all  $k \geq 1$ , we are in the position to coordinate all positions at all successive configurations, using concept names not in  $\Sigma$ .



**Fig. 2.** Enforced bisimulation in lower bound

## DLs with Role Inclusions

By Theorems 5 and 6, it suffices to prove the following.

**Lemma 12.** *Let  $\mathcal{L} \in \text{DL}_{\text{nr}}$  admit role inclusions. It is 2EXPTIME-hard to decide for an  $\mathcal{L}$ -ontology  $\mathcal{O}$ , concept  $C$ , and signature  $\Sigma \subseteq \text{sig}(\mathcal{O})$  whether  $\mathcal{O}, C$  and  $\mathcal{O}, \neg C$  are jointly consistent modulo  $\mathcal{L}(\Sigma)$ -bisimulations.*

As in the proof of Lemma 11, we reduce the word problem for exponentially space bounded ATMs, using the same ATM model as above. In fact the only difference to the proof of Lemma 11 is the way in which we enforce that exponentially many elements are  $\mathcal{L}(\Sigma)$ -bisimilar. We show how to achieve this for  $\mathcal{L} = \mathcal{ALCH}$  using signature  $\Sigma' = (\Sigma \setminus \{r\}) \cup \{r_1, r_2\}$ . The symbols in  $\Sigma' \cap \Sigma$  play exactly the same role as above. The modifications for the inclusion of inverse roles and the universal role are discussed in the appendix. The idea is to construct an ontology  $\mathcal{O}'$  such that  $M$  accepts  $w$  iff  $\mathcal{O}', C$  and  $\mathcal{O}', \neg C$  are jointly consistent modulo  $\mathcal{L}(\Sigma')$ -bisimulations, for  $C = \exists r^n. \top$ .

We replace the nominal  $b$  by an  $r$ -chain of length  $n$  as follows (recall  $r \notin \Sigma'$ ). The ontology  $\mathcal{O}'$  contains the RIs  $r \sqsubseteq r_1, r \sqsubseteq r_2$  and the CI  $\neg \exists r^n. \top \sqcap \exists r_1^n. \top \sqsubseteq R$ . The concept name  $R$  induces a binary tree  $T_R$  of depth  $n$  in which each inner node has an  $r_1$ - and an  $r_2$ -successor, and whose leaves carry counter values from 0 to  $2^n - 1$ , encoded via non- $\Sigma$  concept names. To achieve the bisimilar elements, we use that if there exist models  $\mathcal{I}$  and  $\mathcal{J}$  of  $\mathcal{O}'$  and  $d \in \Delta^{\mathcal{I}}, e \in \Delta^{\mathcal{J}}$  such that

- $d \in (\exists r^n. \top)^{\mathcal{I}}$ ;
- $e \in (\neg \exists r^n. \top)^{\mathcal{J}}$ ;
- $\mathcal{I}, d \sim_{\mathcal{ALCH}, \Sigma'} \mathcal{J}, e$ ;

then it follows that in  $\mathcal{J}$  there exists a binary tree  $T_R$  with root  $e$  and of depth  $n$  such that all leaves of  $T_R$  are  $\mathcal{ALCH}(\Sigma')$ -bisimilar: Due to  $\mathcal{I}, d \sim_{\mathcal{ALCH}, \Sigma'} \mathcal{J}, e$  and  $d \in (\exists r_1^n. \top)^{\mathcal{I}}$ , we have  $e \in (\exists r_1^n. \top)^{\mathcal{J}}$  and thus  $e \in R^{\mathcal{J}}$  which starts  $T_R$ . Moreover, as  $r \sqsubseteq r_i$  for  $i = 1, 2$ , we have for the  $r$ -path starting at  $d$  in  $\mathcal{I}$  for any  $r_1/r_2$  sequence of length  $n$  a corresponding path of length  $n$  starting at  $e$  in  $\mathcal{J}$ . Thus, all leaves of  $T_R$  are  $\mathcal{ALCH}(\Sigma')$ -bisimilar.

The rest of the proof is as above:  $\mathcal{O}'$  enforces that every leaf of  $T_R$  is the start of an infinite tree along role  $s$  along which the same two counters are maintained; the second counter starts at *the value of the counter on the leaf*. The computation tree of  $M$  on input  $w$  is encoded as above and the coordination between consecutive configuration is achieved by the availability of the second counter and using non- $\Sigma$ -symbols.

## 8 Non-Projective $\mathcal{L}$ -Definition Existence

We show that for  $\mathcal{L}$  in  $\text{DL}_{\text{nr}}$ , non-projective  $\mathcal{L}$ -definition existence is in EXPTIME. Note that by Theorem 3 (2),  $\mathcal{ALCCO}$  and  $\mathcal{ALCHCO}$  are the only DLs in  $\text{DL}_{\text{nr}}$  that do not enjoy the BDP. Thus it suffices to consider these two languages.

To motivate our approach, observe that the addition of inverse roles or the universal role to  $\mathcal{ALCCO}$  or  $\mathcal{ALCHCO}$  restores the BDP. The following example

from [12] illustrates what is happening: let  $\mathcal{O}$  be the ontology containing  $A \sqsubseteq \{a\}$ ,  $\{b\} \sqcap B \sqsubseteq \exists r.(\{a\} \sqcap A)$ , and  $\{b\} \sqcap \neg B \sqsubseteq \exists r.(\{a\} \sqcap \neg A)$ . Then  $A$  is explicitly definable under  $\mathcal{O}$  by both  $\{a\} \sqcap \exists r^-. (B \sqcap \{b\})$  and by  $\{a\} \sqcap \exists u. (B \sqcap \{b\})$ , but  $A$  is not explicitly  $\mathcal{ALCO}(\{r, B, b, a\})$ -definable under  $\mathcal{O}$ . This example motivates the following characterization which is proved using Theorem 6 and bisimulation products. Let  $\mathcal{I}, d$  be a pointed model and  $\Sigma$  a signature. Denote by  $\Delta_{\downarrow d, \Sigma}^{\mathcal{I}}$  the smallest subset of  $\Delta^{\mathcal{I}}$  such that  $d \in \Delta_{\downarrow d, \Sigma}^{\mathcal{I}}$  and for all  $(e, e') \in r^{\mathcal{I}}$  with  $r$  a role name in  $\Sigma$ , if  $e \in \Delta_{\downarrow d, \Sigma}^{\mathcal{I}}$ , then  $e' \in \Delta_{\downarrow d, \Sigma}^{\mathcal{I}}$ . The restriction of  $\mathcal{I}$  to  $\Delta_{\downarrow d, \Sigma}^{\mathcal{I}}$  is denoted  $\mathcal{I}_{\downarrow d, \Sigma}$  and called the *interpretation generated by  $d$  w.r.t.  $\Sigma$  in  $\mathcal{I}$* .

**Lemma 13.** *Let  $\mathcal{O}$  be an  $\mathcal{ALCHO}$ -ontology,  $A$  a concept name, and  $\Sigma = \text{sig}(\mathcal{O}) \setminus \{A\}$ . Then  $A$  is not explicitly definable in  $\mathcal{ALCHO}(\Sigma)$  under  $\mathcal{O}$  iff there are pointed models  $\mathcal{I}_1, d$  and  $\mathcal{I}_2, d$  such that*

- $\mathcal{I}_i$  is a model of  $\mathcal{O}$ , for  $i = 1, 2$ ;
- the  $\Sigma$ -reducts of  $\mathcal{I}_{1\downarrow d, \Sigma}$  and  $\mathcal{I}_{2\downarrow d, \Sigma}$  are identical;
- $d \in A^{\mathcal{I}_1}$  and  $d \notin A^{\mathcal{I}_2}$ .

Interpretations  $\mathcal{I}_1$  and  $\mathcal{I}_2$  witnessing that  $A$  is not  $\mathcal{ALCO}(\{r, B, b, a\})$ -definable under the ontology  $\mathcal{O}$  defined above are obtained by taking  $\Delta^{\mathcal{I}_i} = \{a, b\}$  with  $a, b$  interpreting themselves and  $r^{\mathcal{I}_i} = \{(b, a)\}$ , for  $i = 1, 2$ , and setting  $A^{\mathcal{I}_1} = \{a\}$ ,  $B^{\mathcal{I}_1} = \{b\}$ ,  $A^{\mathcal{I}_2} = B^{\mathcal{I}_2} = \emptyset$ . By reformulating the condition given in Lemma 13 as a concept satisfiability problem w.r.t. an  $\mathcal{ALCHO}$ -ontology, we obtain that non-projective  $\mathcal{ALCHO}$ -definition existence is in EXPTIME, as required. The EXPTIME upper bound for  $\mathcal{ALCO}$  follows from the EXPTIME upper bound for  $\mathcal{ALCHO}$  as both languages have the same concept expressions.

## 9 Discussion

We have shown that deciding the existence of interpolants and explicit definitions is 2EXPTIME-complete for DLs ranging from  $\mathcal{ALCO}$  and  $\mathcal{ALCH}$  to  $\mathcal{ALCHIO}^u$ . Our algorithms are not directly applicable in practice to decide the existence of interpolants or explicit definitions nor to compute them if they exist. We are optimistic, however, that the insights from the upper bound proof can be used to design complete tableau-like procedures that extend those in [13]. From a theoretical viewpoint, also many interesting questions remain to be explored. For example, what is the size of interpolants and explicit definitions? The techniques introduced in this paper should be a good starting point. Also, while for  $\mathcal{ALCHIO}$  and logics with the universal role, the 2EXPTIME lower bound holds already under empty ontologies, this remains open for  $\mathcal{ALCO}$  and  $\mathcal{ALCH}$ . Finally, there are many more DLs which do not enjoy the CIP and PBDP and for which the complexity of interpolant and explicit definition existence are open. Examples include expressive languages such as  $\mathcal{SHOIQ}$  and Horn DLs with nominals where recent semantic characterizations [29] might be helpful.

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## A Further Details on Section 2

A few comments regarding the work on interpolant existence for LTL. For the flow of time consisting of the natural numbers, the decidability of interpolant existence in LTL has been established in [45]. Over finite linear orderings decidability was already established in [21, 22]). Note that LTL and Craig interpolation are not mentioned in [45, 21, 22]. Using the fact that regular languages are projectively LTL definable and that LTL and first-order logic are equivalent over the natural numbers, it is easy to see that interpolant existence is the same problem as separability of regular languages in first-order logic, modulo the succinctness of the representation of the inputs.

## B Further Details on Section 4

We give missing details for the proof of Theorem 3.

**Theorem 3.** (1) No  $\mathcal{L} \in \text{DL}_{\text{nr}}$  enjoys the CIP or the PBDP.  
 (2) All  $\mathcal{L} \in \text{DL}_{\text{nr}} \setminus \{\mathcal{ALCCO}, \mathcal{ALCHIO}\}$  enjoy the BDP.  $\mathcal{ALCCO}$  and  $\mathcal{ALCHIO}$  do not enjoy the BDP.

Recall the  $\mathcal{ALCCO}$ -ontology

$$\mathcal{O}_1 = \{\{a\} \sqsubseteq \exists r.\{a\}, A \sqcap \neg\{a\} \sqsubseteq \forall r.(\neg\{a\} \rightarrow \neg A), \\ \neg A \sqcap \neg\{a\} \sqsubseteq \forall r.(\neg\{a\} \rightarrow A)\}.$$

We have argued in the body of the paper that  $\{a\}$  is implicitly definable from  $\Sigma = \{r, A\}$  under  $\mathcal{O}_1$ . Here we show that it is not explicitly  $\mathcal{L}(\Sigma)$ -definable under  $\mathcal{O}_1$  for any  $\mathcal{L} \in \text{DL}_{\text{nr}}$  with nominals: consider the model  $\mathcal{I}$  with  $\Delta^{\mathcal{I}} = \{c, d\}$ ,  $a^{\mathcal{I}} = c$ ,  $r^{\mathcal{I}} = \{(c, c), (c, d), (d, c)\}$ ,  $A^{\mathcal{I}} = \{c, d\}$ . Then  $\mathcal{I}$  is a model of  $\mathcal{O}_1$  and the relation  $S = \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$  is an  $\mathcal{ALCCIO}^u(\Sigma)$ -bisimulation on  $\mathcal{I}$ . Thus, Lemma 1 implies

$$\mathcal{I}, c \equiv_{\mathcal{ALCCIO}^u, \Sigma} \mathcal{I}, d$$

and there is no explicit  $\mathcal{ALCCIO}^u(\Sigma)$ -definition for  $\{a\}$  under  $\mathcal{O}_1$  as any such definition would apply to  $d$  as well.

Similarly, it can be shown that  $\exists r.\top$  cannot be explicitly defined from  $\{r_1, r_2\}$  under  $\mathcal{O}_2$ .

We provide now the proof for Part (2) of the Theorem.

**Proof.** For  $\mathcal{L} \in \text{DL}_{\text{nr}}$  without nominals or with the universal role this follows from Theorems 2.5.4 and 6.2.4 in [11], respectively. To see this observe that modal logics are syntactic variants of descriptions logics and that inverse roles, role inclusions, and the universal role can be introduced as first-order definability conditions on frame classes that are preserved under generated subframes and bisimulation products.

It remains to prove that  $\mathcal{ALCCIO}$  and  $\mathcal{ALCHIO}$  enjoy the BDP. This is done using bisimulation products.

Consider an  $\mathcal{ALCHIO}$ -ontology  $\mathcal{O}$  and let  $A$  be a concept name. Let  $\Sigma = \text{sig}(\mathcal{O}) \setminus \{A\}$ . Assume  $A$  is not explicitly definable from  $\Sigma$  under  $\mathcal{O}$ . By Theorem 6, we find pointed models  $\mathcal{I}_1, d_1$  and  $\mathcal{I}_2, d_2$  such that  $\mathcal{I}_i$  is a model of  $\mathcal{O}$  for  $i = 1, 2$ ,  $d_1 \in A^{\mathcal{I}_1}$ ,  $d_2 \notin A^{\mathcal{I}_2}$ , and  $\mathcal{I}_1, d_1 \sim_{\mathcal{ALCHIO}, \Sigma} \mathcal{I}_2, d_2$ . Take a bisimulation  $S$  witnessing this. We construct an interpretation  $\mathcal{I}$  by taking the *bisimulation product induced by  $S$* : the domain  $\Delta^{\mathcal{I}}$  of  $\mathcal{I}$  is the set of all pairs  $(e_1, e_2) \in S$ . The concept and role names in  $\Sigma$  are interpreted as in cartesian products and a nominal  $a$  in  $\Sigma$  is interpreted as  $(a^{\mathcal{I}_1}, a^{\mathcal{I}_2})$  if  $a$  is in the domain (equivalently, range) of  $S$ . Note that we have projection functions  $f_i : S \rightarrow \Delta^{\mathcal{I}_i}$ , for  $i = 1, 2$ . It is readily checked that the  $f_i$  are  $\mathcal{ALCHIO}(\Sigma)$ -bisimulations between  $\mathcal{I}$  and  $\mathcal{I}_i$ . Moreover, as we have inverse roles, the image of  $S$  under  $f_i$  is a maximal connected component of  $\mathcal{I}_i$  in the sense that if  $(e, e') \in r^{\mathcal{I}_i}$  and  $e \in f_i(S)$  or  $e' \in f_i(S)$  then  $e' \in f_i(S)$  or  $e \in f_i(S)$ , respectively. We now define interpretations  $\mathcal{J}_1$  and  $\mathcal{J}_2$  as the interpretation  $\mathcal{I}$  except that  $A^{\mathcal{J}_i} = f_i^{-1}(A^{\mathcal{I}_i})$ , for  $i = 1, 2$ . Then the  $f_i$  are  $\mathcal{ALCHIO}(\Sigma \cup \{A\})$ -bisimulations between  $\mathcal{J}_i$  and  $\mathcal{I}_i$ , for  $i = 1, 2$ . Note, however, that the  $\mathcal{J}_i$  do not necessarily interpret all nominals, as for a nominal  $\{a\}$ , the element  $a^{\mathcal{I}_1}$  might be in a different connected component than  $d_1$  (equivalently:  $a^{\mathcal{I}_2}$  is in a different component than  $d_2$ ). To address this, let  $\mathcal{I}'$  be the restriction of  $\mathcal{I}_1$  to  $\Delta^{\mathcal{I}_1} \setminus \text{dom}(S)$ . Then, obtain  $\mathcal{J}'_i$  by taking the disjoint union of  $\mathcal{J}_i$  and  $\mathcal{I}'$ ,  $i = 1, 2$ . One can now prove that  $\mathcal{J}'_1$  and  $\mathcal{J}'_2$  are models of  $\mathcal{O}$  whose  $\Sigma$ -reducts coincide and such that  $(d_1, d_2) \in A^{\mathcal{J}'_1}$  but  $(d_1, d_2) \notin A^{\mathcal{J}'_2}$ . It follows that  $A$  is not implicitly definable using  $\Sigma$  under  $\mathcal{O}$ , as required.  $\square$

## C Proofs for Section 5

The following example illustrates explicit definability of nominals.

*Example 14.* Consider the  $\mathcal{ALCHIO}$  ontology  $\mathcal{O}$ , about detectives and spies, that consists of the following CIs ( $\mathcal{O}$  is a variant of an ontology introduced in [46]):

$$\begin{aligned} \mathcal{O} = \{ & \exists \text{suspects.} \top \sqsubseteq \text{Detective}, \text{Detective} \sqsubseteq \forall \text{deceives.} \perp, \\ & \text{Detective} \sqsubseteq \neg \text{Spy}, \text{Detective} \equiv \{d_1\} \sqcup \{d_2\} \sqcup \{d_3\}, \\ & \{s_1\} \sqsubseteq \neg \text{Spy}, \{s_4\} \sqsubseteq \text{Spy}, \{s_1\} \sqsubseteq \exists \text{deceives.} \{s_2\}, \\ & \{s_2\} \sqsubseteq \exists \text{deceives.} \{s_3\}, \{s_3\} \sqsubseteq \exists \text{deceives.} \{s_4\}, \\ & \{s_4\} \sqsubseteq \forall \text{deceives}^- . \text{Spy}, \{d_1\} \sqsubseteq \forall \text{suspects.} \{s_1\}, \\ & \{d_3\} \sqsubseteq \forall \text{suspects.} \{s_4\}, \\ & \{d_2\} \sqsubseteq \exists \text{suspects.} \{s_2\} \sqcap \exists \text{suspects.} \{s_3\} \}. \end{aligned}$$

Reasoning by cases, it can be seen that

$$\mathcal{O} \models \{d_2\} \equiv \exists \text{suspects.} (\text{Spy} \sqcap \exists \text{deceives}^- . \neg \text{Spy}),$$

thus, for  $\Sigma = \{\text{Spy}, \text{suspects}, \text{deceives}\}$ , there is an explicit  $\mathcal{ALCHIO}(\Sigma)$  definition of  $\{d_2\}$  under  $\mathcal{O}$ . Another definition of  $\{d_2\}$  under  $\mathcal{O}$  is given by  $\exists \text{suspects.} \{s_2\} \sqcap \exists \text{suspects.} \{s_3\}$ . On the other hand, for  $\Sigma' = \{\text{suspects}\}$ , there does not exist any explicit  $\mathcal{ALCHIO}(\Sigma')$  definition of  $\{d_2\}$  under  $\mathcal{O}$ .

**Theorem 5.** *Let  $\mathcal{L} \in \text{DL}_{\text{nr}}$ . Let  $\mathcal{O}_1, \mathcal{O}_2$  be  $\mathcal{L}$ -ontologies and let  $C_1, C_2$  be  $\mathcal{L}$ -concepts, and  $\Sigma = \text{sig}(\mathcal{O}_1, C_1) \cap \text{sig}(\mathcal{O}_2, C_2)$ . Then the following conditions are equivalent:*

1. *there is no  $\mathcal{L}$ -interpolant for  $C_1 \sqsubseteq C_2$  under  $\mathcal{O}_1 \cup \mathcal{O}_2$ ;*
2.  *$\mathcal{O}_1 \cup \mathcal{O}_2, C_1$  and  $\mathcal{O}_1 \cup \mathcal{O}_2, \neg C_2$  are jointly consistent modulo  $\mathcal{L}(\Sigma)$ -bisimulations.*

**Proof.** “1  $\Rightarrow$  2”. Assume there is no  $\mathcal{L}$ -interpolant for  $C_1 \sqsubseteq C_2$  under  $\mathcal{O}_1 \cup \mathcal{O}_2$ . Let  $\Sigma = \text{sig}(\mathcal{O}_1, C_1) \cap \text{sig}(\mathcal{O}_2, C_2)$  and define

$$\Gamma = \{D \mid \mathcal{O}_1 \cup \mathcal{O}_2 \models C_1 \sqsubseteq D, D \in \mathcal{L}(\Sigma)\}.$$

Then  $\mathcal{O}_1 \cup \mathcal{O}_2 \not\models D \sqsubseteq C_2$ , for any  $D \in \Gamma$ . As  $\Gamma$  is closed under conjunction and by compactness (recall that  $\mathcal{ALCHIO}^u$  is a fragment of first-order logic), there exists a model  $\mathcal{J}$  of  $\mathcal{O}_1 \cup \mathcal{O}_2$  and a node  $d$  in it such that  $d \in D^{\mathcal{J}}$  for all  $D \in \Gamma$  but  $d \notin C_2^{\mathcal{J}}$ . Consider the full  $\Sigma$ -type  $t_{\mathcal{J}}^{\mathcal{L}(\Sigma)}(d)$  of  $d$  in  $\mathcal{J}$ , defined as the set of all  $\mathcal{L}(\Sigma)$ -concepts  $D$  such that  $d \in D^{\mathcal{J}}$ . Then by compactness there exists a model  $\mathcal{I}$  of  $\mathcal{O}_1 \cup \mathcal{O}_2$  and a node  $e$  in it such that  $e \in C_1^{\mathcal{I}}$  and  $e \in D^{\mathcal{I}}$  for all  $D \in t_{\mathcal{J}}^{\mathcal{L}(\Sigma)}(d)$ . Thus  $\mathcal{I}, e \equiv_{\mathcal{L}, \Sigma} \mathcal{J}, d$ . For every interpretation  $\mathcal{I}$  there exists an  $\omega$ -saturated elementary extension  $\mathcal{I}'$  of  $\mathcal{I}$  [15]. Thus, it follows from the fact that  $\mathcal{ALCHIO}^u$  is a fragment of first-order logic that we may assume that both  $\mathcal{I}$  and  $\mathcal{J}$  are  $\omega$ -saturated. By Lemma 1,  $\mathcal{I}, e \sim_{\mathcal{L}, \Sigma} \mathcal{J}, d$ .

“2  $\Rightarrow$  1”. Assume an  $\mathcal{L}$ -interpolant  $D$  for  $C_1 \sqsubseteq C_2$  under  $\mathcal{O}_1 \cup \mathcal{O}_2$  exists. Assume that Condition 2 holds, that is, there are models  $\mathcal{I}_1$  and  $\mathcal{I}_2$  of  $\mathcal{O}_1 \cup \mathcal{O}_2$  and  $d_i \in \Delta^{\mathcal{I}_i}$  for  $i = 1, 2$  such that  $d_1 \in C_1^{\mathcal{I}_1}$  and  $d_2 \notin C_2^{\mathcal{I}_2}$  and  $\mathcal{I}_1, d_1 \sim_{\mathcal{L}, \Sigma} \mathcal{I}_2, d_2$ , where  $\Sigma = \text{sig}(\mathcal{O}_1, C_1) \cap \text{sig}(\mathcal{O}_2, C_2)$ . Then, by Lemma 1,  $\mathcal{I}_1, d_1 \equiv_{\mathcal{L}, \Sigma} \mathcal{I}_2, d_2$ . But then from  $d_1 \in C_1^{\mathcal{I}_1}$  we obtain  $d_1 \in D^{\mathcal{I}_1}$  and so  $d_2 \in D^{\mathcal{I}_2}$  which implies  $d_2 \in C_2^{\mathcal{I}_2}$ , a contradiction.  $\square$

## D Proofs for Section 6

**Lemma 9.** *Let  $\mathcal{L} \in \text{DL}_{\text{nr}}$ . Assume  $\mathcal{O}_1, \mathcal{O}_2$  are  $\mathcal{L}$ -ontologies,  $C_1, C_2$  are  $\mathcal{L}$ -concepts, and let  $\Sigma \subseteq \text{sig}(\Xi)$  be a signature. The following conditions are equivalent:*

1.  *$\mathcal{O}_1, C_1$  and  $\mathcal{O}_2, C_2$  are jointly consistent modulo  $\mathcal{L}(\Sigma)$ -bisimulations.*
2. *there exists an  $\mathcal{L}(\Sigma)$ -good set  $\mathcal{S}$  and  $\Xi$ -types  $t_1, t_2$  with  $C_1 \in t_1$  and  $C_2 \in t_2$  such that  $t_1 \in T_1$  and  $t_2 \in T_2$  for some  $(T_1, T_2) \in \mathcal{S}$ .*

**Proof.** “1  $\Rightarrow$  2”. Let  $\mathcal{I}_1, d_1 \sim_{\mathcal{L}, \Sigma} \mathcal{I}_2, d_2$  for models  $\mathcal{I}_1$  of  $\mathcal{O}_1$  and  $\mathcal{I}_2$  of  $\mathcal{O}_2$  such that  $d_1, d_2$  realize  $\Xi$ -types  $t_1, t_2$  and  $C_1 \in t_1, C_2 \in t_2$ . Define  $\mathcal{S}$  by setting  $(T_1, T_2) \in \mathcal{S}$  if there is  $d \in \Delta^{\mathcal{I}_i}$  for some  $i \in \{1, 2\}$  such that

$$T_j = \{\text{tp}_{\Xi}(\mathcal{I}_j, e) \mid e \in \Delta^{\mathcal{I}_j}, \mathcal{I}_i, d \sim_{\mathcal{L}, \Sigma} \mathcal{I}_j, e\},$$

for  $j = 1, 2$ . We then say that  $(T_1, T_2)$  is induced by  $d$  in  $\mathcal{I}_i$ . It is straightforward to show that  $\mathcal{S}$  is  $\mathcal{L}(\Sigma)$ -good and satisfies Point 2.  $\mathcal{S}$  is good for nominals: as

$\text{tp}_{\Xi}(\mathcal{I}_i, a^{\mathcal{I}_i})$  is the only  $\Xi$ -type containing a nominal  $a$  that is realized in  $\mathcal{I}_i$ , for  $i = 1, 2$ , there exists exactly one  $t_a^i$  with  $\{a\} \in t_a^i \in \bigcup_{(T_1, T_2) \in \mathcal{S}} T_i$ . Moreover, the only pair  $(T_1, T_2) \in \mathcal{S}$  with  $t_a^i \in T_i$  is the one induced by  $a^{\mathcal{I}_i}$  in  $\mathcal{I}_i$ . Clearly, if  $a \in \Sigma$ , then by Condition [AtomI] for  $\mathcal{L}(\Sigma)$ -bisimulations, that pair either takes the form  $(\{t_a^1\}, \{t_a^2\})$  or the form  $(\{t_a^1\}, \emptyset)$  and  $(\emptyset, \{t_a^2\})$ , respectively.  $\mathcal{S}$  is good for  $\mathcal{O}_1, \mathcal{O}_2$  by definition.  $\mathcal{S}$  is  $\Sigma$ -concept name coherent by Condition [AtomC] of  $\mathcal{L}(\Sigma)$ -bisimulations. For existential saturation, assume first that  $\mathcal{L}$  does not have inverse roles and let  $(T_1, T_2) \in \mathcal{S}$  and  $\exists r. C \in t' \in T_i$ . Then  $(T_1, T_2)$  is induced by some  $d_{t'}$  in  $\mathcal{I}_i$  realizing  $t'$ . There exists  $f_{t'} \in \Delta^{\mathcal{I}_i}$  such that  $(d_{t'}, f_{t'}) \in r^{\mathcal{I}_i}$  and  $f_{t'} \in C^{\mathcal{I}_i}$ . Let  $(T'_1, T'_2) \in \mathcal{S}$  be induced by  $f_{t'}$ . Condition (1) holds since  $t' \rightsquigarrow_{r, \mathcal{O}_i} \text{tp}_{\Xi}(\mathcal{I}_i, f_{t'})$ . For Condition (2) assume that  $\mathcal{O}_i \models r \sqsubseteq s$  with  $s \in \Sigma$ . Then  $(T_1, T_2) \rightsquigarrow_s (T'_1, T'_2)$  follows from the Condition [Forth] of  $\mathcal{L}(\Sigma)$ -bisimulations. If  $\mathcal{L}$  has inverse roles, then Condition (2I) holds for  $(T'_1, T'_2)$  by the definition of  $\mathcal{L}(\Sigma)$ -bisimulations; and if  $\mathcal{L}$  has the universal role, then Condition (3u) follows by the definition of  $\mathcal{L}(\Sigma)$ -bisimulations.

“2  $\Rightarrow$  1”. Assume  $\mathcal{S}$  is  $\mathcal{L}(\Sigma)$ -good and we have  $\Xi$ -types  $s_1, s_2$  with  $C_1 \in s_1$  and  $C_2 \in s_2$  such that  $s_1 \in S_1$  and  $s_2 \in S_2$  for some  $(S_1, S_2) \in \mathcal{S}$ .

If  $\mathcal{L}$  does not admit inverse roles, then we construct interpretations  $\mathcal{I}_1$  and  $\mathcal{I}_2$  by setting

$$\begin{aligned} \Delta^{\mathcal{I}_i} &:= \{(t, (T_1, T_2)) \mid t \in T_i \text{ and } (T_1, T_2) \in \mathcal{S}\} \\ r^{\mathcal{I}_i} &:= \{((t, p), (t', p')) \in \Delta^{\mathcal{I}_i} \times \Delta^{\mathcal{I}_i} \mid t \rightsquigarrow_{r, \mathcal{O}_i} t', \\ &\quad \forall s \in \Sigma ((\mathcal{O}_i \models r \sqsubseteq s) \Rightarrow p \rightsquigarrow_s p')\} \\ A^{\mathcal{I}_i} &:= \{(t, p) \in \Delta^{\mathcal{I}_i} \mid A \in t\} \\ a^{\mathcal{I}_i} &:= (t, (T_1, T_2)) \in \Delta^{\mathcal{I}_i}, \{a\} \in t \in T_i \end{aligned}$$

If  $\mathcal{L}$  admits inverse roles then replace ‘ $s \in \Sigma$ ’ by ‘ $s$  a role over  $\Sigma$ ’ and ‘ $p \rightsquigarrow_s p'$ ’ by ‘ $p \rightsquigarrow_s p'$ ’ in the definition of  $r^{\mathcal{I}_i}$ .

We first show by induction that for  $i = 1, 2$  and all  $D \in \Xi$  and  $(t, p) \in \Delta^{\mathcal{I}_i}$ :  $D \in t$  iff  $(t, p) \in D^{\mathcal{I}_i}$ . The only interesting step is for  $D = \exists r. D'$  and  $D \in t$ . Then by existential saturation there exists  $(T'_1, T'_2) \in \mathcal{S}$  such that (1) there exists  $t' \in T'_i$  with  $D' \in t'$  and  $t \rightsquigarrow_{r, \mathcal{O}_i} t'$  and (2) if  $\mathcal{O}_i \models r \sqsubseteq s$  with  $s \in \Sigma$ , then  $p \rightsquigarrow_s (T'_1, T'_2)$  (we only consider the case without inverse roles, the case with inverse roles is similar). Thus, by definition,  $((t, p), (t', (T'_1, T'_2))) \in r^{\mathcal{I}_i}$  and, by induction hypothesis,  $(t', (T'_1, T'_2)) \in D'^{\mathcal{I}_i}$ . Hence  $(t, p) \in D^{\mathcal{I}_i}$ , as required.

Thus  $\mathcal{I}_i$  satisfies the CIs in  $\mathcal{O}_i$  for  $i = 1, 2$ .  $\mathcal{I}_i$  also satisfies the RIs in  $\mathcal{O}_i$ : this follows from the observation that  $t \rightsquigarrow_{r, \mathcal{O}_i} t'$  implies  $t \rightsquigarrow_{s, \mathcal{O}_i} t'$  and  $p \rightsquigarrow_r p'$  implies  $p \rightsquigarrow_s p'$  whenever  $\mathcal{O}_i \models r \sqsubseteq s$ . Now let

$$S = \{((t_1, p_1), (t_2, p_2)) \in \Delta^{\mathcal{I}_1} \times \Delta^{\mathcal{I}_2} \mid p_1 = p_2\}.$$

Then  $S$  is a  $\mathcal{L}(\Sigma)$ -bisimulation between  $\mathcal{I}_1$  and  $\mathcal{I}_2$  witnessing that  $\mathcal{O}_1, C_1$  and  $\mathcal{O}_2, C_2$  are jointly consistent modulo  $\mathcal{L}(\Sigma)$ -bisimulations: to show this observe that [AtomI] is satisfied as  $\mathcal{S}$  is good for nominals. [AtomC] is satisfied by  $\Sigma$ -concept coherence. The conditions [Forth] and [Back] are satisfied by the interpretation of the role names and the definition of  $p \rightsquigarrow_r p'$ . The additional

conditions for inverse roles and the universal roles correspond to the additional conditions on good sets  $\mathcal{S}$ .  $\square$

**Lemma 10.** *Let  $\mathcal{L} \in \text{DL}_{\text{nr}}$ . Then it is decidable in double exponential time whether for  $\mathcal{L}$ -ontologies  $\mathcal{O}_1, \mathcal{O}_2$ ,  $\mathcal{L}$ -concepts  $C_1, C_2$ , and a signature  $\Sigma \subseteq \text{sig}(\Xi)$  there exists an  $\mathcal{S}$  and  $t_1, t_2$  satisfying Condition 2 of Lemma 9.*

**Proof.** We start with  $\text{ALCOH}$ . Assume  $\mathcal{O}_1, \mathcal{O}_2, C_1, C_2$ , and  $\Sigma$  are given. We can enumerate in double exponential time the maximal sets  $\mathcal{U} \subseteq 2^{T(\Xi)} \times 2^{T(\Xi)}$  that are good for nominals and for  $\mathcal{O}_1, \mathcal{O}_2$ : simply remove from  $2^{T(\Xi)} \times 2^{T(\Xi)}$  all  $(T_1, T_2)$  such that  $T_i$  contains a  $t$  that is not satisfiable in any model of  $\mathcal{O}_i$  and then take the sets  $\mathcal{U}$  in which for each nominal  $a \in \text{sig}(\Xi)$  the types  $t_a^i$  and the pairs  $(T_1, T_2)$  in which  $t_a^i$  occurs have been selected, for  $i = 1, 2$ . Also make sure that either  $(\{t_a^1\}, \{t_a^2\}) \in \mathcal{U}$  or  $(\{t_a^1\}, \emptyset), (\emptyset, \{t_a^2\}) \in \mathcal{U}$ , for  $a \in \Sigma$ . Then we eliminate from any such  $\mathcal{U}$  recursively all pairs that are not  $\Sigma$ -concept name coherent or not existentially saturated. Let  $\mathcal{S}_0 \subseteq \mathcal{U}$  be the largest fixpoint of this procedure. Then one can easily show that there exists a set  $\mathcal{S}$  satisfying Condition 2 of Lemma 9 iff there exists a maximal set  $\mathcal{U}$  that is good for nominals and  $\mathcal{O}_1, \mathcal{O}_2$  such that the largest fixpoint  $\mathcal{S}_0$  satisfies Condition 2 of Lemma 9. The elimination procedure is in double exponential time.

The modifications needed for the remaining DLs are straightforward: for DLs with inverse roles modify the recursive elimination procedure by considering Condition 2 $\mathcal{I}$  for existential saturation and for DLs with the universal role remove any  $(T_1, T_2)$  with  $T_i = \emptyset$  for  $i = 1$  or  $i = 2$  from any  $\mathcal{U}$ .  $\square$

## E Proofs for Section 7

We reduce the word problem for exponentially space bounded alternating Turing machines (ATMs). We actually use a slightly unusual ATM model which is easily seen to be equivalent to the standard model.

An *alternating Turing machine (ATM)* is a tuple  $M = (Q, \Theta, \Gamma, q_0, \Delta)$  where  $Q = Q_{\exists} \uplus Q_{\forall}$  is the set of states that consists of *existential states* in  $Q_{\exists}$  and *universal states* in  $Q_{\forall}$ . Further,  $\Theta$  is the input alphabet and  $\Gamma$  is the tape alphabet that contains a *blank symbol*  $\square \notin \Theta$ ,  $q_0 \in Q_{\exists}$  is the *starting state*, and the *transition relation*  $\Delta$  is of the form  $\Delta \subseteq Q \times \Gamma \times Q \times \Gamma \times \{L, R\}$ . The set  $\Delta(q, a) := \{(q', a', M) \mid (q, a, q', a', M) \in \Delta\}$  must contain exactly two or zero elements for every  $q \in Q$  and  $a \in \Gamma$ . Moreover, the state  $q'$  must be from  $Q_{\forall}$  if  $q \in Q_{\exists}$  and from  $Q_{\exists}$  otherwise, that is, existential and universal states alternate. Note that there is no accepting state. The ATM accepts if it runs forever and rejects otherwise. Starting from the standard ATM model, this can be achieved by assuming that exponentially space bounded ATMs terminate on any input and then modifying them to enter an infinite loop from the accepting state.

More formally, a *configuration* of an ATM is a word  $wqw'$  with  $w, w' \in \Gamma^*$  and  $q \in Q$ . We say that  $wqw'$  is *existential* if  $q$  is, and likewise for *universal*. *Successor*

*configurations* are defined in the usual way. Note that every configuration has exactly two successor configurations.

A *computation tree* of an ATM  $M$  on input  $w$  is an infinite tree whose nodes are labeled with configurations of  $M$  such that

- the root is labeled with the initial configuration  $q_0w$ ;
- if a node is labeled with an existential configuration  $wqw'$ , then it has a single successor which is labeled with a successor configuration of  $wqw'$ ;
- if a node is labeled with a universal configuration  $wqw'$ , then it has two successors which are labeled with the two successor configurations of  $wqw'$ .

An ATM  $M$  *accepts* an input  $w$  if there is a computation tree of  $M$  on  $w$ .

### E.1 DLs with Nominals

We reduce the word problem for  $2^n$ -space bounded ATMs which is well-known to be 2EXPTIME-hard [14]. We first provide the reduction for  $\mathcal{ALCO}$  using an ontology  $\mathcal{O}$  and signature  $\Sigma$  such that  $\mathcal{O}$  contains concept names that are not in  $\Sigma$ . We set

$$\Sigma = \{r, s, Z, B_{\forall}, B_{\exists}^1, B_{\exists}^2\} \cup \{A_{\sigma} \mid \sigma \in \Gamma \cup (Q \times \Gamma)\}.$$

The idea of the reduction is as follows. The ontology  $\mathcal{O}$  enforces that  $r(b, b)$  holds using the CI  $\{b\} \sqsubseteq \exists r.\{b\}$ . Moreover, any node distinct from  $b$  with an  $r$ -successor lies on an infinite  $r$ -path  $\rho$ . Along  $\rho$ , a counter counts modulo  $2^n$  using concept names not in  $\Sigma$ . Additionally, in each point of  $\rho$  starts an infinite tree along role  $s$  that is supposed to mimick the computation tree of  $M$ . Along this tree, two counters are maintained:

- one counter starting at 0 and counting modulo  $2^n$  to divide the tree in subpaths of length  $2^n$ ; each such path of length  $2^n$  represents a configuration;
- another counter starting at *the value of the counter on  $\rho$*  and also counting modulo  $2^n$ .

To link successive configurations we use that if there exist models  $\mathcal{I}$  and  $\mathcal{J}$  of  $\mathcal{O}$  such that  $\mathcal{I}, b^{\mathcal{I}} \sim_{\mathcal{ALCO}, \Sigma} \mathcal{J}, d$  for some  $d \neq b^{\mathcal{J}}$  it follows that in  $\mathcal{J}$  all nodes on some  $r$ -path  $\rho$  through  $d$  are  $\mathcal{ALCO}(\Sigma)$ -bisimilar. Thus, each node on  $\rho$  is the starting point of  $s$ -trees with identical  $\Sigma$ -decorations. As on the  $m$ th  $s$ -tree the second counter starts at all nodes at distances  $k \times 2^n - m$ , for all  $k \geq 1$ , we are in the position to coordinate all positions at all successive configurations.

The ontology  $\mathcal{O}$  is constructed as follows. We enforce that any node  $d$  that does not equal  $b$  and has an  $r$ -successor satisfies a concept name  $I_s$  that triggers an  $A_i$ -counter along the role name  $r$  and starts  $s$ -trees. We thus have for concept names  $A_i, \bar{A}_i, i < n$ :

$$\begin{aligned}
\neg\{b\} \sqcap \exists r. \top &\sqsubseteq I_s \\
I_s &\sqsubseteq \prod_{i < n} (A_i \sqcup \bar{A}_i) \\
I_s &\sqsubseteq \exists r. \top \sqcap \forall r. I_s \\
A_i \sqcap \prod_{j < i} A_j &\sqsubseteq \forall r. \bar{A}_i \\
\bar{A}_i \sqcap \prod_{j < i} A_j &\sqsubseteq \forall r. A_i \\
A_i \sqcap \bigsqcup_{j < i} \bar{A}_j &\sqsubseteq \forall r. A_i \\
\bar{A}_i \sqcap \bigsqcup_{j < i} \bar{A}_j &\sqsubseteq \forall r. \bar{A}_i
\end{aligned}$$

Using the concept names  $I_s$ , we start the  $s$ -trees with two counters, realized using concept names  $U_i, \bar{U}_i$  and  $V_i, \bar{V}_i$ ,  $i < n$ , and initialized to 0 and the value of the  $A$ -counter, respectively:

$$\begin{aligned}
I_s &\sqsubseteq (U = 0) \\
I_s \sqcap A_j &\sqsubseteq V_j && j < n \\
I_s \sqcap \bar{A}_j &\sqsubseteq \bar{V}_j && j < n \\
\top &\sqsubseteq \exists s. \top
\end{aligned}$$

Here,  $(U = 0)$  is an abbreviation for the concept  $\prod_{i=1}^n \bar{U}_i$ , we use similar abbreviations below. The counters  $U_i$  and  $V_i$  are incremented along  $s$  analogously to how  $A_i$  is incremented along  $r$ , so we omit details. Configurations of  $M$  are represented between two consecutive points having  $U$ -counter value 0. We next enforce the structure of the computation tree, assuming that  $q_0 \in Q_\forall$ :

$$\begin{aligned}
I_s &\sqsubseteq B_\forall && (\dagger) \\
(U < 2^n - 1) \sqcap B_\forall &\sqsubseteq \forall s. B_\forall \\
(U < 2^n - 1) \sqcap B_\exists^i &\sqsubseteq \forall s. B_\exists^i && i \in \{1, 2\} \\
(U = 2^n - 1) \sqcap B_\forall &\sqsubseteq \forall s. (B_\exists^1 \sqcup B_\exists^2) \\
(U = 2^n - 1) \sqcap (B_\exists^1 \sqcup B_\exists^2) &\sqsubseteq \forall s. B_\forall \\
(U = 2^n - 1) \sqcap B_\forall &\sqsubseteq \exists s. Z \sqcap \exists s. \neg Z
\end{aligned}$$

These sentences enforce that all points which represent a configuration satisfy exactly one of  $B_\forall, B_\exists^1, B_\exists^2$  indicating the kind of configuration and, if existential, also a choice of the transition function. The symbol  $Z \in \Sigma$  enforces the branching.

We next set the initial configuration, for input  $w = a_0, \dots, a_{n-1}$ .

$$\begin{aligned}
I_s &\sqsubseteq A_{q_0, a_0} \\
I_s &\sqsubseteq \forall s^k . A_{a_k} && 0 < k < n \\
I_s &\sqsubseteq \forall s^{n+1} . \text{Blank} \\
\text{Blank} &\sqsubseteq A_{\square} \\
\text{Blank} \sqcap (U < 2^n - 1) &\sqsubseteq \forall s . \text{Blank}
\end{aligned}$$

To coordinate consecutive configurations, we associate with  $M$  functions  $f_i$ ,  $i \in \{1, 2\}$  that map the content of three consecutive cells of a configuration to the content of the middle cell in the  $i$ -the successor configuration (assuming an arbitrary order on the set  $\Delta(q, a)$ ). In what follows, we ignore the cornercases that occur at the border of configurations; they are treated in a similar way. Clearly, for each possible such triple  $(\sigma_1, \sigma_2, \sigma_3) \in \Gamma \cup (Q \times \Gamma)$ , there is an  $\mathcal{ALC}$  concept  $C_{\sigma_1, \sigma_2, \sigma_3}$  which is true at an element  $a$  of the computation tree iff  $a$  is labeled with  $A_{\sigma_1}$ ,  $a$ 's  $s$ -successor  $b$  is labeled with  $A_{\sigma_2}$ , and  $b$ 's  $s$ -successor  $c$  is labeled with  $A_{\sigma_3}$ . Now, in each configuration, we synchronize elements with  $V$ -counter 0 by including for every  $\sigma = (\sigma_1, \sigma_2, \sigma_3)$  and  $i \in \{1, 2\}$  the following sentences:

$$\begin{aligned}
(V = 2^n - 1) \sqcap (U < 2^n - 2) \sqcap C_{\sigma_1, \sigma_2, \sigma_3} \\
&\sqsubseteq \forall s . A_{f_1(\sigma)}^1 \sqcap \forall s . A_{f_2(\sigma)}^2 \\
(V = 2^n - 1) \sqcap (U < 2^n - 2) \sqcap C_{\sigma_1, \sigma_2, \sigma_3} \sqcap B_{\exists}^i \\
&\sqsubseteq \forall s . A_{f_i(\sigma)}^i
\end{aligned}$$

The concept names  $A_{\sigma}^i$  are used as markers (not in  $\Sigma$ ) and are propagated along  $s$  for  $2^n$  steps, exploiting the  $V$ -counter. The superscript  $i \in \{1, 2\}$  determines the successor configuration that the symbol is referring to. After crossing the end of a configuration, the symbol  $\sigma$  is propagated using concept names  $A'_{\sigma}$  (the superscript is not needed anymore because the branching happens at the end of the configuration, based on  $Z$ ).

$$\begin{aligned}
(U < 2^n - 1) \sqcap A_{\sigma}^i &\sqsubseteq \forall s . A_{\sigma}^i \\
(U = 2^n - 1) \sqcap B_{\forall} \sqcap A_{\sigma}^1 &\sqsubseteq \forall s . (\neg Z \sqcup A'_{\sigma}) \\
(U = 2^n - 1) \sqcap B_{\forall} \sqcap A_{\sigma}^2 &\sqsubseteq \forall s . (Z \sqcup A'_{\sigma}) \\
(U = 2^n - 1) \sqcap B_{\exists}^i \sqcap A_{\sigma}^i &\sqsubseteq \forall s . A'_{\sigma} && i \in \{1, 2\} \\
(V < 2^n - 1) \sqcap A'_{\sigma} &\sqsubseteq \forall s . A'_{\sigma} \\
(V = 2^n - 1) \sqcap A'_{\sigma} &\sqsubseteq \forall s . A_{\sigma}
\end{aligned}$$

For those  $(q, a)$  with  $\Delta(q, a) = \emptyset$ , we add the concept inclusion

$$A_{q, a} \sqsubseteq \perp.$$

The following lemma establishes correctness of the reduction.

**Lemma 15.** *The following conditions are equivalent:*

1.  $M$  accepts  $w$ .
2. there exist models  $\mathcal{I}$  and  $\mathcal{J}$  of  $\mathcal{O}$  such that  $\mathcal{I}, b^{\mathcal{I}} \sim_{\mathcal{ALCO}, \Sigma} \mathcal{J}, d$ , for some  $d \neq b^{\mathcal{J}}$ .

**Proof.** “1  $\Rightarrow$  2”. If  $M$  accepts  $w$ , there is a computation tree of  $M$  on  $w$ . We construct a single interpretation  $\mathcal{I}$  with  $\mathcal{I}, b^{\mathcal{I}} \sim_{\mathcal{ALCO}, \Sigma} \mathcal{I}, d$  for some  $d \neq b^{\mathcal{I}}$  as follows. Let  $\widehat{\mathcal{J}}$  be the infinite tree-shaped interpretation that represents the computation tree of  $M$  on  $w$  as described above, that is, configurations are represented by sequences of  $2^n$  elements linked by role  $s$  and labeled by  $B_{\forall}, B_{\exists}^1, B_{\exists}^2$  depending on whether the configuration is universal or existential, and in the latter case the superscript indicates which choice has been made for the existential state. Finally, the first element of the first successor configuration of a universal configuration is labeled with  $Z$ . Observe that  $\widehat{\mathcal{J}}$  interprets only the symbols in  $\Sigma$  as non-empty. Now, we obtain interpretation  $\mathcal{I}_k, k < 2^n$  from  $\widehat{\mathcal{J}}$  by interpreting non- $\Sigma$ -symbols as follows:

- the root of  $\mathcal{I}_k$  satisfies  $I_s$ ;
- the  $U$ -counter starts at 0 at the root and counts modulo  $2^n$  along each  $s$ -path;
- the  $V$ -counter starts at  $k$  at the root and counts modulo  $2^n$  along each  $s$ -path;
- the auxiliary concept names of the shape  $A_{\sigma}^i$  and  $A'_{\sigma}$  are interpreted in a minimal way so as to satisfy the concept inclusions listed above. Note that the respective concept inclusions are Horn, hence there is no choice.

Now obtain  $\mathcal{I}$  from  $\widehat{\mathcal{J}}$  and the  $\mathcal{I}_k$  by creating an infinite outgoing  $r$ -path  $\rho$  from some node  $d$  (with the corresponding  $A$ -counter) and adding  $\mathcal{I}_k, k < 2^n$  to every node with  $A$ -counter value  $k$  on the  $r$ -path, identifying the roots of the  $\mathcal{I}_k$  with the node on the path. Additionally, add  $\widehat{\mathcal{J}}$  to  $b^{\mathcal{I}} = b$  by identifying  $b$  with the root of  $\widehat{\mathcal{J}}$ . It should be clear that  $\mathcal{I}$  is as required. In particular, the reflexive and symmetric closure of

- all pairs  $(b, e), (e, e')$ , with  $e, e'$  on  $\rho$ , and
- all pairs  $(e, e'), (e', e'')$ , with  $e$  in  $\widehat{\mathcal{J}}$  and  $e', e''$  copies of  $e$  in the trees  $\mathcal{I}_k$ .

is an  $\mathcal{ALCO}(\Sigma)$ -bisimulation  $S$  on  $\mathcal{I}$  with  $(b, d) \in S$ .

“2  $\Rightarrow$  1”. Assume that  $\mathcal{I}, b^{\mathcal{I}} \sim_{\mathcal{ALCO}, \Sigma} \mathcal{J}, d$  for some  $d \neq b^{\mathcal{J}}$ . As argued above, due to the  $r$ -self loop at  $b^{\mathcal{I}}$ , from  $d$  there has to be an outgoing infinite  $r$ -path on which all  $s$ -trees are  $\mathcal{ALCO}(\Sigma)$ -bisimilar. Since  $\mathcal{I}$  is a model of  $\mathcal{O}$ , all these  $s$ -trees are additionally labeled with some auxiliary concept names not in  $\Sigma$ , depending on the distance from their roots on  $\rho$ . Using the concept inclusions in  $\mathcal{O}$  and the arguments given in their description, it can be shown that all  $s$ -trees contain a computation tree of  $M$  on input  $w$  (which is solely represented with concept names in  $\Sigma$ ).  $\square$

The same ontology  $\mathcal{O}$  can be used for the remaining DLs with nominals. For  $\mathcal{ALCCO}^u$ , exactly the same proof works. For the DLs with inverse roles the infinite  $r$ -path  $\rho$  has to be extended to an infinite  $r^-$ -path.

Using the ontology  $\mathcal{O}$  defined above we define a new ontology  $\mathcal{O}'$  to obtain the 2EXPTIME lower bound for signatures  $\Sigma' = \text{sig}(\mathcal{O}') \setminus \{b\}$ . Fix a role name  $r_E$  for any concept name  $E \in \text{sig}(\mathcal{O}) \setminus \Sigma$ . Now replace in  $\mathcal{O}$  any occurrence of  $E \in \text{sig}(\mathcal{O}) \setminus \Sigma$  by  $\exists r_E.\{b\}$  and denote the resulting ontology by  $\mathcal{O}'$ .

**Lemma 16.** *The following conditions are equivalent:*

1.  $M$  accepts  $w$ .
2. there exist models  $\mathcal{I}$  and  $\mathcal{J}$  of  $\mathcal{O}'$  such that  $\mathcal{I}, b^{\mathcal{I}} \sim_{\mathcal{ALCCO}, \Sigma'} \mathcal{J}, d$ , for some  $d \neq b^{\mathcal{J}}$ .

**Proof.** “1  $\Rightarrow$  2”. We modify the interpretation  $\mathcal{I}$  defined in the proof of Lemma 15 in such a way that we obtain a model of  $\mathcal{O}'$  and such that the  $\mathcal{ALCCO}(\Sigma)$ -bisimulation  $S$  on  $\mathcal{I}$  defined in that proof is, in fact, an  $\mathcal{ALCCO}(\Sigma')$ -bisimulation on the new interpretation. Formally, obtain  $\mathcal{I}'$  from  $\mathcal{I}$  by interpreting every  $r_E$ ,  $E \in \text{sig}(\mathcal{O}) \setminus \Sigma$  as follows:

- (i) there is an  $r_E$ -edge from  $e$  to  $b^{\mathcal{I}'}$  for all  $e \in E^{\mathcal{I}'}$ ;
- (ii) there is an  $r_E$ -edge from  $e$  to all nodes on the path  $\rho$  for all  $(e, e') \in S$  and  $e' \in E^{\mathcal{I}'}$ ;
- (iii) there are no more  $r_E$ -edges.

Note that, by (i),  $\mathcal{I}'$  is a model of  $\mathcal{O}'$ . By (ii), the relation  $S$  defined in the proof of Lemma 15 is an  $\mathcal{ALCCO}(\Sigma')$ -bisimulation. In particular, by (i), elements  $e' \in E^{\mathcal{I}'}$  have now an  $r_E$ -edge to  $b^{\mathcal{I}'}$ , so any element  $e$  bisimilar to  $e'$ , that is,  $(e, e') \in S$ , needs an  $r_E$ -successor to some element bisimilar to  $b^{\mathcal{I}'}$ . Since all elements on the path  $\rho$  are bisimilar to  $b^{\mathcal{I}'}$ , these  $r_E$ -successors exist due to (ii).

“2  $\Rightarrow$  1”. This direction remains the same as in the proof of Lemma 15.  $\square$

The extension to our DLs with inverse roles and the universal role is again straightforward.

## E.2 DLs with role inclusions

By Theorems 5 and 6, it suffices to prove the following result.

**Lemma 17.** *Let  $\mathcal{L} \in \text{DL}_{\text{nr}}$  admit role inclusions. It is 2EXPTIME-hard to decide for an  $\mathcal{L}$ -ontology  $\mathcal{O}$ , concept  $C$ , and signature  $\Sigma \subseteq \text{sig}(\mathcal{O})$  whether  $\mathcal{O}, C$  and  $\mathcal{O}, \neg C$  are jointly consistent modulo  $\mathcal{L}(\Sigma)$ -bisimulations.*

As in the proof of Lemma 11, we reduce the word problem for exponentially space bounded alternating Turing machines (ATMs). We use the same ATM model as above.

We first provide the reduction for  $\mathcal{ALCH}$ . Set

$$\Sigma = \{r_1, r_2, s, Z, B_{\forall}, B_{\exists}^1, B_{\exists}^2\} \cup \{A_{\sigma} \mid \sigma \in \Gamma \cup (Q \times \Gamma)\}.$$

The symbols  $s, Z, B_\forall, B_{\exists}^1, B_{\exists}^2$  and  $A_\sigma, \sigma \in \Gamma \cup (Q \times \Gamma)$ , play exactly the same role as above. The main difference is that we replace the nominal  $b$  by an  $r$ -chain of length  $n$ . The ontology  $\mathcal{O}$  contains the RIs  $r \sqsubseteq r_1, r \sqsubseteq r_2$  and the CI  $\neg\exists r^n.\top \sqcap \exists r_1^n.\top \sqsubseteq R$ .  $R$  is the root of a binary tree  $T_R$  of depth  $n$  in which each node has  $r_1$  and  $r_2$ -successors. The leaves of  $T_R$  carry counter values from 0 to  $2^n - 1$ . Moreover, in each leaf of  $T_R$  starts an infinite tree along role  $s$  that mimicks the computation tree of  $M$  (thus the leaves of  $T_R$  play the role of the path  $\rho$ ). Along these trees we maintain the same two counters as before:

- one counter starting at 0 and counting modulo  $2^n$  to divide the tree in subpaths of length  $2^n$ ; each such path of length  $2^n$  represents a configuration;
- another counter starting at *the value of the counter on the leaf* and also counting modulo  $2^n$ .

To link successive configurations we use that if there exist models  $\mathcal{I}$  and  $\mathcal{J}$  of  $\mathcal{O}$  and  $d \in \Delta^{\mathcal{I}}, e \in \Delta^{\mathcal{J}}$  such that

- $d \in (\exists r^n.\top)^{\mathcal{I}}$ ;
- $e \in (\neg\exists r^n.\top)^{\mathcal{J}}$ ;
- $\mathcal{I}, d \sim_{\mathcal{ALC}, \Sigma} \mathcal{J}, e$ ;

then it follows that in  $\mathcal{J}$  there exists a binary tree  $T_R$  with root  $e$  and of depth  $n$  such that all leaves of  $T_R$  are  $\mathcal{ALC}(\Sigma)$ -bisimilar: Due to  $\mathcal{I}, d \sim_{\mathcal{ALC}, \Sigma} \mathcal{J}, e$  and  $d \in (\exists r_1^n.\top)^{\mathcal{I}}$ , we have  $e \in (\exists r_1^n.\top)^{\mathcal{J}}$  and thus  $e \in R^{\mathcal{I}}$  which starts  $T_R$ . Moreover, as  $r \sqsubseteq r_i$  for  $i = 1, 2$ , we have for the  $r$ -path starting at  $d$  in  $\mathcal{I}$  for any  $r_1/r_2$  sequence of length  $n$  a corresponding path of length  $n$  starting at  $e$  in  $\mathcal{J}$ . Thus, each leaf of  $T_R$  is  $\Sigma$ -bisimilar to the element reached after  $n$ -steps from  $d$ . Hence, each leaf is the starting point of  $s$ -trees with identical  $\Sigma$ -decorations.

In detail, let  $w = a_0, \dots, a_{n-1}$  be an input to  $M$  of length  $n$ . The ontology  $\mathcal{O}$  is constructed as follows. We enforce that any node satisfying  $R$  generates a binary tree of depth  $n$  whose leaves satisfy a concept name  $L_R$  and also satisfy the canonical value of an  $A$ -counter along the role names  $r_1, r_2$  and start  $s$ -trees. We thus have for concept names  $A_i, \bar{A}_i, i < n$ :

$$\begin{aligned}
R &\sqsubseteq R_0 \\
R_i &\sqsubseteq \forall r_1.R_{i+1} \sqcap \forall r_2.R_{i+1} \\
R_n &\sqsubseteq L_R \\
R_i &\sqsubseteq \exists r_1.A_i \sqcap \forall r_1.A_i \\
R_i &\sqsubseteq \exists r_2.\bar{A}_i \sqcap \forall r_2.\bar{A}_i \\
A_i &\sqsubseteq \forall r_1.A_i \sqcap \forall r_2.A_i \\
\bar{A}_i &\sqsubseteq \forall r_1.\bar{A}_i \sqcap \forall r_2.\bar{A}_i
\end{aligned}$$

Using the concept name  $L_R$ , we start the  $s$ -trees with two counters, realized using concept names  $U_i, \bar{U}_i$  and  $V_i, \bar{V}_i, i < n$ , and initialized to 0 and the value

of the  $A$ -counter, respectively:

$$\begin{aligned}
L_R &\sqsubseteq \prod_{i < n} \overline{U}_i \\
L_R \sqcap A_j &\sqsubseteq V_j & j < n \\
L_R \sqcap \overline{A}_j &\sqsubseteq \overline{V}_j & j < n \\
&\top \sqsubseteq \exists s. \top
\end{aligned}$$

The structure of the computation tree, the initial configuration, and the coordination between consecutive configurations is done using the same concept inclusions as in the proof of Lemma 11, starting from inclusion (†) and replacing  $I_s$  with  $L_R$ . We can then prove the following very similar to Lemma 15.

**Lemma 18.** *The following conditions are equivalent:*

1.  $M$  accepts  $w$ .
2. there exist models  $\mathcal{I}$  and  $\mathcal{J}$  of  $\mathcal{O}$  such that  $\mathcal{I}, d \sim_{\mathcal{ALCH}, \Sigma} \mathcal{J}, e$ , for some  $d \in (\exists r^n. \top)^{\mathcal{I}}$  and  $e \notin (\exists r^n. \top)^{\mathcal{J}}$ .

**Proof.** “1  $\Rightarrow$  2”. If  $M$  accepts  $w$ , there is a computation tree of  $M$  on  $w$ . We construct a single interpretation  $\mathcal{I}$  with  $\mathcal{I}, d \sim_{\mathcal{ALCH}, \Sigma} \mathcal{I}, e$  for some  $d, e$  with  $d \in (\exists r^n. \top)^{\mathcal{I}}$  and  $e \notin (\exists r^n. \top)^{\mathcal{I}}$  as follows. Let  $\widehat{\mathcal{J}}$  be the infinite tree-shaped interpretation that represents the computation tree of  $M$  on  $w$  as described above, that is, configurations are represented by sequences of  $2^n$  elements linked by role  $s$  and labeled by  $B_{\forall}, B_{\exists}^1, B_{\exists}^2$  depending on whether the configuration is universal or existential, and in the latter case the superscript indicates which choice has been made for the existential state. Finally, the first element of the first successor configuration of a universal configuration is labeled with  $Z$ . Observe that  $\widehat{\mathcal{J}}$  interprets only the symbols in  $\Sigma$  as non-empty. Now, we obtain interpretation  $\mathcal{I}_k, k < 2^n$  from  $\widehat{\mathcal{J}}$  by interpreting non- $\Sigma$ -symbols as follows:

- the root of  $\mathcal{I}_k$  satisfies  $L_R$ ;
- the  $U$ -counter starts at 0 at the root and counts modulo  $2^n$  along each  $s$ -path;
- the  $V$ -counter starts at  $k$  at the root and counts modulo  $2^n$  along each  $s$ -path;
- the auxiliary concept names of the shape  $A_{\sigma}^i$  and  $A'_{\sigma}$  are interpreted in a minimal way so as to satisfy the concept inclusions that enforce the coordination between consecutive configurations (c.f. the concept inclusions in proof of Lemma 11). Note that the respective concept inclusions are Horn, hence there is no choice.

Now obtain  $\mathcal{I}$  from  $\widehat{\mathcal{J}}$  and the  $\mathcal{I}_k$  as follows: First, create a path of length  $n$  from some node  $d$  so that consecutive elements are connected with  $r, r_1, r_2$ , and identify the end of the path with the root of  $\widehat{\mathcal{J}}$ . Then create the binary tree  $T_R$  (as described above) with root  $e$  and identify the leaf having  $A$ -counter value  $k$  with the root of  $\mathcal{I}_k$ , for all  $k < 2^n$ .  $\mathcal{I}$  is as required since, by construction,  $d \in (\exists r^n. \top)^{\mathcal{I}}, e \notin (\exists r^n. \top)^{\mathcal{I}}$ , and the reflexive and symmetric closure of

- all pairs  $(d', e'), (e', e'')$  such that  $d'$  has distance  $\ell < n$  from  $d$  and  $e', e''$  have distance  $\ell$  from  $e$ , and
- all pairs  $(b, b'), (b', b'')$ , with  $b$  in  $\widehat{\mathcal{J}}$  and  $b', b''$  copies of  $b$  in the trees  $\mathcal{I}_k$ .

is an  $\mathcal{ALCH}(\Sigma)$ -bisimulation  $S$  on  $\mathcal{I}$  with  $(d, e) \in S$ .

“2  $\Rightarrow$  1”. Assume that  $\mathcal{I}, d \sim_{\mathcal{ALCH}, \Sigma} \mathcal{J}, e$  for models  $\mathcal{I}, \mathcal{J}$  of  $\mathcal{O}$  and some  $d, e$  with  $d \in (\exists r^n. \top)^{\mathcal{I}}$  and  $e \notin (\exists r^n. \top)^{\mathcal{J}}$ . As argued above, there is a binary tree  $T_R$  starting at  $e$  whose leaves are  $\mathcal{ALCH}(\Sigma)$ -similar. In particular, all these leaves root  $s$ -trees which are  $\mathcal{ALCH}(\Sigma)$ -bisimilar. Since  $\mathcal{J}$  is a model of  $\mathcal{O}$ , all these  $s$ -trees are additionally labeled with some auxiliary concept names not in  $\Sigma$ , depending on the value of the  $A$ -counter of the corresponding leaf. Using the concept inclusions in  $\mathcal{O}$  and the arguments given in their description, it can be shown that all  $s$ -trees contain a computation tree of  $M$  on input  $w$  (which is solely represented with concept names in  $\Sigma$ ).  $\square$

It remains to note that exactly the same proof works as well for  $\mathcal{ALCH}\mathcal{I}$  and  $\mathcal{ALCH}\mathcal{I}^u$ , as the relation  $S$  constructed in the direction 1  $\Rightarrow$  2 above is actually an  $\mathcal{ALCH}\mathcal{I}^u(\Sigma)$ -bisimulation.

## F Proofs for Section 8

**Lemma 13.** *Let  $\mathcal{O}$  be an  $\mathcal{ALCH}\mathcal{O}$ -ontology,  $A$  a concept name, and  $\Sigma = \text{sig}(\mathcal{O}) \setminus \{A\}$ . Then  $A$  is not explicitly definable in  $\mathcal{ALCH}\mathcal{O}(\Sigma)$  under  $\mathcal{O}$  iff there are pointed models  $\mathcal{I}_1, d$  and  $\mathcal{I}_2, d$  such that*

- $\mathcal{I}_i$  is a model of  $\mathcal{O}$ , for  $i = 1, 2$ ;
- the  $\Sigma$ -reducts of  $\mathcal{I}_{1 \downarrow d, \Sigma}$  and  $\mathcal{I}_{2 \downarrow d, \Sigma}$  are identical;
- $d \in A^{\mathcal{I}_1}$  and  $d \notin A^{\mathcal{I}_2}$ .

**Proof.** Clearly, if the conditions of Lemma 13 hold, then  $A$  is not explicitly definable from  $\Sigma$  under  $\mathcal{O}$ , by Theorem 6. Conversely, assume  $A$  is not explicitly definable from  $\Sigma$  under  $\mathcal{O}$ . By Theorem 6, we find pointed models  $\mathcal{I}_1, d_1$  and  $\mathcal{I}_2, d_2$  such that  $\mathcal{I}_i$  is a model of  $\mathcal{O}$  for  $i = 1, 2$ ,  $d_1 \in A^{\mathcal{I}_1}$ ,  $d_2 \notin A^{\mathcal{I}_2}$ , and  $\mathcal{I}_1, d_1 \sim_{\mathcal{ALCH}\mathcal{O}, \Sigma} \mathcal{I}_2, d_2$ . Take a bisimulation  $S$  witnessing this. We may assume that  $S$  is a bisimulation between the set  $\Delta_{\downarrow d_1, \Sigma}^{\mathcal{I}_1}$  generated by  $d_1$  and  $\Sigma$  in  $\mathcal{I}_1$  and the set  $\Delta_{\downarrow d_2, \Sigma}^{\mathcal{I}_2}$  generated by  $d_2$  and  $\Sigma$  in  $\mathcal{I}_2$ . We construct an interpretation  $\mathcal{I}$  by taking the bisimulation product induced by  $S$  (see proof of Theorem 3 (2)). Note that we have projection functions  $f_i : S \rightarrow \Delta^{\mathcal{I}_i}$  which are  $\mathcal{ALCO}(\Sigma)$ -bisimulations between  $\mathcal{I}$  and  $\mathcal{I}_i$ . However, as in the proof of Theorem 3 (2),  $\mathcal{I}$  does not necessarily interpret all nominals. We address this in the following.

Let  $\mathcal{J}_i$  be the restriction of  $\mathcal{I}_i$  to  $\Delta^{\mathcal{I}_1} \setminus \Delta_{\downarrow d_i, \Sigma}^{\mathcal{I}_i}$ , for  $i = 1, 2$ . We now define interpretations  $\mathcal{J}'_1$  and  $\mathcal{J}'_2$  as follows:  $\mathcal{J}'_i$  is the disjoint union of  $\mathcal{I}$  and  $\mathcal{J}_i$  extended by

- adding to the interpretation of  $A$  all nodes in  $f_i^{-1}(A^{\mathcal{I}_i})$ ;
- adding  $(e, (e_1, e_2))$  to the interpretation of  $r$  if  $e \in \Delta^{\mathcal{I}_i} \setminus \Delta_{\downarrow d_i, \Sigma}^{\mathcal{I}_i}$ ,  $e_i \in \Delta_{\downarrow d_i, \Sigma}^{\mathcal{I}_i}$ , and  $(e, e_i) \in r^{\mathcal{I}_i}$ .

$\mathcal{J}'_1$  and  $\mathcal{J}'_2$  with the node  $(d_1, d_2)$  are as required.  $\square$

Let  $\mathcal{O}$  be an  $\mathcal{ALCH}\mathcal{O}$ -ontology,  $A$  a concept name, and  $\Sigma = \text{sig}(\mathcal{O}) \setminus \{A\}$ . Take concept names  $D$  (for the domain of the interpretation  $\mathcal{I}'_i$  generated by  $d$  w.r.t.  $\Sigma$  in  $\mathcal{I}_i$ ),  $D_i$  (for the domain of  $\mathcal{I}_i$ ),  $i = 1, 2$ , and a copy  $A'$  of  $A$  and fresh copies  $a'$  of the nominals  $a$  in  $\Sigma$ . Let  $\mathcal{O}'$  be the ontology obtained from  $\mathcal{O}$  by replacing  $A$  by  $A'$  and all nominals  $a$  in  $\mathcal{O}$  by  $a'$ . Let  $\mathcal{O}(D_1)$  be the relativization of  $\mathcal{O}$  to  $D_1$  and let  $\mathcal{O}'(D_2)$  be the relativization of  $\mathcal{O}'$  to  $D_2$ . Now consider:

$$\begin{aligned} \mathcal{O}'' = & \mathcal{O}(D_1) \cup \mathcal{O}'(D_2) \cup \\ & \{D \sqsubseteq \forall r.D \mid r \in \Sigma\} \cup \\ & \{D \sqsubseteq D_1, D \sqsubseteq D_2\} \cup \\ & \{\{a\} \sqsubseteq D_1 \mid a \in \Sigma\} \cup \\ & \{\{a'\} \sqsubseteq D_2 \mid a \in \Sigma\} \cup \\ & \{D \sqcap \{a\} \sqsubseteq \{a'\} \mid a \in \Sigma\} \cup \\ & \{D \sqcap \{a'\} \sqsubseteq \{a\} \mid a \in \Sigma\} \end{aligned}$$

Then the conditions of Lemma 13 hold iff the concept  $A \sqcap \neg A' \sqcap D$  is satisfiable w.r.t. the ontology  $\mathcal{O}''$ .