Properties of Conservative Extensions and Modules

C. Lutz, U. Sattler, and F. Wolter

July 31, 2008

Modules

- A module is a part of a system which functions independently from the system. The connection between the module and the system is provided by an interface.
- What does it mean that a logical theory functions independently from a system? What is an interface?

Definition

Here, a logical theory is a black box which provides answers to queries about some vocabulary Σ of interest. The interface is given by a query language $\mathcal{QL}$ and a signature Σ.

Interfaces: arithmetic

Let T be some logical theory of arithmetic over the signature $\{+, \times, s, <, 0\}$. We use abbreviations $1 = s0$, $2 = ss0$, etc., and n, m, k range over these number.

Possible interfaces $(\mathcal{QL}, \Sigma)$ are:

- Pre-school arithmetic: $\mathcal{QL} = \{n + m = k \mid n, m, k \leq 20\}$, $\Sigma = \{0, s, +\}$;
- Primary school: $\mathcal{QL} = \{n + m = k, n \times m = k\}$, $\Sigma = \{0, s, +, \times\}$;
- Undergraduate: $\mathcal{QL} =$ linear equations , $\Sigma = \{+, \times, 0, s\}$;
- Mathematician: $\mathcal{QL} =$ Diophantine equations, $\Sigma = \{+, \times, 0, s\}$;
- Logician: $\mathcal{QL} = \mathrm{SO}$, $\Sigma = \{+, \times, 0, s, f_1, f_2, \ldots\}$;

Every user/application (system) requires its own interface (view, set of observables) of a module.

Interfaces: ontologies

Let T be a TBox (defining terms of some medical domain). Possible interfaces $(\mathcal{QL}, \Sigma)$:

- Secretary:
  $\mathcal{QL} =$ all inclusions $A \sqsubseteq B$, where A, B are concept names,
  $\Sigma =$ Predicates relevant to hospital administration;
- Researcher:
  $\mathcal{QL} =$ all inclusions $A \sqsubseteq B$, where A, B are concept names,
  $\Sigma =$ Predicates relevant to cancer research;
- Terminologist: $\mathcal{QL} =$ all $\mathcal{ALC}$-concept inclusions,
  $\Sigma =$ Predicates relevant to anatomy;
- Terminologist who does not know his language: $\mathcal{QL} =$ SO,
  $\Sigma =$ Predicates relevant to hospital administration.

Interface for querying instance data

Ontology provides background theory when querying instance data.

Query language $\mathcal{QL}$: $\mathcal{A} \rightarrow q$, where $\mathcal{A}$ represents instance data and q is a query.

Example: Instance data $\mathcal{A}$:

$$\{\text{Country}(\text{France}), \text{Country}(\text{Columbia}), \ldots, \text{LocatedinEurope}(\text{France}), \ldots\}$$

Query: $q = \text{EuropeanCountry}(\text{France})$.
Then

$$T \models \mathcal{A} \rightarrow q$$

if $T \models \text{Country} \sqcap \text{LocatedinEurope} \sqsubseteq \text{EuropeanCountry}$.

When can one module be equivalently replaced by another module

SO denotes the set of sentences of Second-Order Logic.

Definition

Let T_1, T_2 be finite sets of SO-sentences, $\mathcal{QL} \subseteq$ SO a query language, and Σ a signature. Then T_1 and T_2 are Σ-inseparable w.r.t. $\mathcal{QL}$, in symbols

$$T_1 \equiv_S^{\mathcal{QL}} T_2,$$

if for all $\varphi \in \mathcal{QL}$ with $\mathrm{sig}(\varphi) \subseteq \Sigma$:

$$T_1 \models \varphi \Leftrightarrow T_2 \models \varphi.$$

Inseparability w.r.t. SO

Theorem

Let T_1 and T_2 be finite sets of SO-sentences and Σ a signature. Then the following are equivalent:

- $T_1 \equiv_\Sigma^{SO} T_2$;
- $\{M_{|\Sigma} \mid M \models T_1\} = \{M_{|\Sigma} \mid M \models T_2\}$.

Point 1 implies Point 2. Suppose $M \models T_1$, but there does not exist $M' \models T_2$ with $M'_{|\Sigma} = M_{|\Sigma}$. Then

$$M \not\models \exists P_1 \cdots \exists P_n . \bigwedge T_2,$$

where $\{P_1, \ldots, P_n\} = \mathrm{sig}(T_2) \setminus \Sigma$. Hence

- $T_2 \models \exists P_1 \cdots \exists P_n . \bigwedge T_2$;
- $T_1 \not\models \exists P_1 \cdots \exists P_n . \bigwedge T_2$.

Conservativity and inseparability: FO

Let $T_1 \subseteq T_2$ be finite sets FO-sentences. Then

- T_2 is a deductive Σ-conservative extension of T_1 iff T_1 and T_2 are Σ-inseparable w.r.t. FO;
- T_2 is a model Σ-conservative extension of T_1 iff T_1 and T_2 are Σ-inseparable w.r.t. SO.

"$\Leftarrow$ for Point 2".
Follows from the equivalence of

- $T_1 \equiv_\Sigma^{SO} T_2$;
- $\{M_{|\Sigma} \mid M \models T_1\} = \{M_{|\Sigma} \mid M \models T_2\}$.

Conservativity and inseparability

Let $\mathcal{L}$ be a description logic. Then $\mathcal{QL}_{\mathcal{L}}$ (or just $\mathcal{L}$)denotes the set of concept inclusion $C \sqsubseteq D$, where C, D are $\mathcal{L}$-concepts.

Let $T_1 \subseteq T_2$ be TBoxes. Then

- T_2 is a deductive Σ-conservative extension of T_1 in $\mathcal{ALC}$ iff T_1 and T_2 are Σ-inseparable w.r.t. $\mathcal{QL}_{\mathcal{ALC}}$;
- T_2 is a model Σ-conservative extension of T_1 iff T_1 and T_2 are Σ-inseparable w.r.t. SO.

Theorem

Still: Σ-inseparability of $\mathcal{ALC}$-TBoxes w.r.t. $\mathcal{ALC}$ is 2ExpTime-complete.

Characterisation of Σ-inseparability w.r.t. $\mathcal{ALC}$

We will use the following extension of the characterisation from Day 2:

Theorem

Let T_1 and T_2 be $\mathcal{ALC}$-TBoxes and Σ a signature. Then T_1 and T_2 are Σ-inseparable w.r.t. $\mathcal{ALC}$ iff

- *for all models $(\mathcal{I}_1, d_1)$ of T_1 there exists model $(\mathcal{I}_2, d_2)$ of T_2 with*
$$(\mathcal{I}_1, d_1) \sim_\Sigma (\mathcal{I}_2, d_2).$$
- *vice versa.*

Programme:

- Properties of Σ-inseparability and the connection to Interpolation Property.
- Module extraction: Given $(T, \mathcal{QL}, \Sigma)$, extract a smallest $T' \subseteq T$ such that
$$T' \equiv_{\Sigma}^{\mathcal{QL}} T.$$
- Forgetting: Given $(T, \mathcal{QL}, \Sigma)$, compute a T' such that $\mathrm{sig}(T') \subseteq \Sigma$ and
$$T' \equiv_{\Sigma}^{\mathcal{QL}} T.$$
- Theory versioning (diff): if $T_1 \not\equiv_{\Sigma}^{\mathcal{QL}} T_2$, give an informative description of the difference between T_1 and T_2.

Properties of $\equiv_{\Sigma}^{\mathcal{QL}}$

Simple properties of $\equiv_{\Sigma}^{\mathcal{QL}}$

$\mathcal{QL}$ set of SO-sentences, $\mathcal{L} \subseteq$ SO, and T_1, T_2 range over finite sets of $\mathcal{L}$-sentences:

- $\equiv_{\Sigma}^{\mathcal{QL}}$ is an equivalence relation.
- If $\Sigma \subseteq \Sigma'$, then

$$T_1 \equiv_{\Sigma'}^{\mathcal{QL}} T_2 \quad \Rightarrow \quad T_1 \equiv_{\Sigma}^{\mathcal{QL}} T_2.$$

- If $\mathcal{QL} \subseteq \mathcal{QL}'$, then

$$T_1 \equiv_{\Sigma}^{\mathcal{QL}'} T_2 \quad \Rightarrow \quad T_1 \equiv_{\Sigma}^{\mathcal{QL}} T_2.$$

Robustness properties

Let $\mathcal{L}, \mathcal{QL} \subseteq \mathrm{SO}$. Then $(\mathcal{L}, \mathcal{QL})$ is *robust*

- under vocabulary extensions if, for all finite sets T_1 and T_2 of $\mathcal{L}$-sentences and signatures Σ, Σ' with $\Sigma' \cap \mathrm{sig}(T_1 \cup T_2) \subseteq \Sigma$, the following holds:

$$T_1 \equiv_{\Sigma}^{\mathcal{QL}} T_2 \quad \Rightarrow \quad T_1 \equiv_{\Sigma'}^{\mathcal{QL}} T_2.$$

- under joins if, for all finite sets T_1 and T_2 of $\mathcal{L}$-sentences and signatures Σ with $\mathrm{sig}(T_1) \cap \mathrm{sig}(T_2) \subseteq \Sigma$, the following holds for $i = 1, 2$:

$$T_1 \equiv_{\Sigma}^{\mathcal{QL}} T_2 \quad \Rightarrow \quad T_i \equiv_{\Sigma}^{\mathcal{QL}} T_1 \cup T_2.$$

- under replacement if, for all all finite sets T, T_1 and T_2 of $\mathcal{L}$-sentences and signatures Σ with $\mathrm{sig}(T) \cap \mathrm{sig}(T_1 \cup T_2) \subseteq \Sigma$, the following holds:

$$T_1 \equiv_{\Sigma}^{\mathcal{QL}} T_2 \quad \Rightarrow \quad T_1 \cup T \equiv_{\Sigma}^{\mathcal{QL}} T_2 \cup T.$$

Concept hierarchy

Let $\mathcal{QL}_C$ denote the set of implications $A \sqsubseteq B$ such that A, B are concept names.

Theorem

$(\mathcal{ALC}, \mathcal{QL}_C)$ is robust under vocabulary extensions, but not under joins nor replacement.

$$T_1 = \{A \sqsubseteq \exists r.B\}, \quad T_2 = \{\exists r.B \sqsubseteq E\}, \quad \Sigma = \{r, A, B, E\}.$$

Then

- $T_1 \equiv_\Sigma^{\mathcal{QL}_C} T_2$;
- $T_1 \cup T_2 \not\equiv_\Sigma^{\mathcal{QL}_C} T_2$.
- $T_1 \cup \{B \equiv \bot\} \not\equiv_\Sigma^{\mathcal{QL}_C} T_2 \cup \{B \equiv \bot\}$.

Basic observations

Theorem

Let $\mathcal{L} \subseteq \mathcal{QL}$. If $(\mathcal{QL}, \mathcal{QL})$ is robust under vocabulary extensions (joins, replacement), then $(\mathcal{L}, \mathcal{QL})$ is robust under vocabulary extensions (joins, replacement).

Robustness under joins: Suppose

$$T_1 \equiv_{\Sigma}^{\mathcal{QL}} T_2 \quad \Rightarrow \quad T_i \equiv_{\Sigma}^{\mathcal{QL}} T_1 \cup T_2,$$

for all finite sets T_1 and T_2 of $\mathcal{QL}$-sentences. Then, because $\mathcal{L} \subseteq \mathcal{QL}$,

$$T_1 \equiv_{\Sigma}^{\mathcal{QL}} T_2 \quad \Rightarrow \quad T_i \equiv_{\Sigma}^{\mathcal{QL}} T_1 \cup T_2,$$

for all finite sets T_1 and T_2 of $\mathcal{L}$-sentences.

Basic observations

Theorem

(SO,SO) is robust under vocabulary extensions, joins and replacement.

Robustness under joins: suppose

$$\{M_{|\Sigma} \mid M \models T_1\} = \{M_{|\Sigma} \mid M \models T_2\}$$

and $\mathrm{sig}(T_1) \cap \mathrm{sig}(T_2) \subseteq \Sigma$. Then every $M_{|\Sigma}$ with $M \models T_1$ can be expanded to a model M' of T_2. Moreover, we may assume that the interpretation of $\mathrm{sig}(T_1)$ in M and M' is the same. But then $M' \models T_1 \cup T_2$ and $M'_{|\Sigma} = M_\Sigma$. Hence

$$\{M_{|\Sigma} \mid M \models T_1\} = \{M_{|\Sigma} \mid M \models T_1 \cup T_2\}.$$

Robustness under Vocabulary Extensions and Interpolation

Definition

A query language $\mathcal{QL}$ has weak interpolation iff for every finite set T of $\mathcal{QL}$-sentences and $\mathcal{QL}$-sentence φ such that $T \models \varphi$ there exists a set $I(T,\varphi)$ of $\mathcal{QL}$-sentence such that

- $\mathrm{sig}(I(T,\varphi)) \subseteq \mathrm{sig}(T) \cap \mathrm{sig}(\varphi)$;
- $T \models I(T,\varphi)$;
- $I(T,\varphi) \models \varphi$.

$\mathcal{QL}$ has interpolation if there always exists a finite set $I(T,\varphi)$ with these properties.

Many results on many distinct versions of interpolation in logic and software specification!

Weak Interpolation implies R. under Vocabulary Extensions

Theorem

If $\mathcal{QL}$ has weak interpolation, then $(\mathcal{QL}, \mathcal{QL})$ is robust under vocabulary extensions.

Proof. Suppose $T_1 \equiv_{\Sigma}^{\mathcal{QL}} T_2$. Let $\varphi \in \mathcal{QL}$ with $sig(\varphi) \cap sig(T_1 \cup T_2) \subseteq \Sigma$ such that $T_1 \models \varphi$. We show $T_2 \models \varphi$. By weak interpolation,

$$T_1 \models I(T_1, \varphi) \models \varphi.$$

From $sig(I(T_1, \varphi)) \subseteq \Sigma$

$$T_2 \models I(T_1, \varphi).$$

Hence $T_2 \models \varphi$.

R. under Vocabulary Extensions implies Weak Interpolation

Theorem

Suppose $(\mathcal{QL}, \mathcal{QL})$ is robust under vocabulary extensions for possibly infinite sets of sentences. Then $\mathcal{QL}$ has weak interpolation.

Proof. Assume $T \models \varphi$. Set $\Sigma = \text{sig}(T) \cap \text{sig}(\varphi)$ and

$$T_\Sigma = \{\psi \in \mathcal{QL} \mid T \models \psi, \text{sig}(\psi) \subseteq \Sigma\}.$$

Then

- T and T_Σ are Σ-inseparable w.r.t. $\mathcal{QL}$.

By robustness under vocabulary extensions,

- T and T_Σ are Σ'-inseparable w.r.t. $\mathcal{QL}$, for $\Sigma' = \text{sig}(\varphi)$.

From $T \models \varphi$ we obtain $T_\Sigma \models \varphi$. Hence $I(T, \varphi) = T_\Sigma$ is a weak interpolant.

Interpolation

The following languages have interpolation (and are robust under vocabulary extensions):

- FO;
- $\mathcal{ALC}$, $\mathcal{ALCQ}$, $\mathcal{ALCI}$, $\mathcal{ALCQI}$.

Proof.

- FO: Logic Textbooks;
- DLs: Konev, Lutz, Walther, Wolter: Formal Properties of Modularization, 2008.

Counterexamples: nominals

$\mathcal{ALCO}$ extends $\mathcal{ALC}$ by concept names i, j, etc. interpreted as singleton sets.

Theorem

$\mathcal{ALCO}$ does not have weak interpolation and even $(\mathcal{ALC}, \mathcal{ALCO})$ is not robust under vocabulary extensions.

Let

$$T_1 = \{\top \sqsubseteq \exists r.\top\}, \quad T_2 = T_1 \cup \{A \sqsubseteq \forall r.\neg A, \neg A \sqsubseteq \forall r.A\}.$$

$T_1 \equiv_\Sigma^{\mathcal{ALCO}} T_2$ for $\Sigma = \{r\}$.
Observe $i \sqsubseteq \forall r.\neg i$ separates the two TBoxes w.r.t. $\mathcal{QL}_{\mathcal{ALCO}}$, for any nominal i.
Thus, $T_1 \not\equiv_{\Sigma'}^{\mathcal{ALCO}} T_2$ for $\Sigma' = \Sigma \cup \{i\}$.

Counterexample: role hierarchies

$\mathcal{ALC}_H$ extends $\mathcal{ALC}$ by axioms of the form $r \sqsubseteq s$, for roles r, s.

Theorem

$(\mathcal{ALC}_H, \mathcal{ALC}_H)$ is not robust under vocabulary extensions.

Let

$$T_1 = \{\top \sqsubseteq \forall r_i \forall r_j.\bot \mid i,j = 1,2\} \cup \{\exists r_1.\top \equiv \exists r_2.\top\},$$

$$T_2 = T_1 \cup \{s \sqsubseteq r_1, s \sqsubseteq r_2, \exists r_1.\top \sqsubseteq \exists s.\top\}.$$

Then $T_1 \equiv_\Sigma^{\mathcal{ALC}_H} T_2$ for $\Sigma = \{r_1, r_2\}$.
$\exists r_1.\top \sqcap \forall r_1.A \sqsubseteq \exists r_2.A$ separates the two ontologies, where A is a fresh concept name.

FO on finite models

Theorem

FO *on finite models has weak interpolation but no interpolation.*

Weak Interpolation:
Suppose $T \models_f \varphi$. Let $\Sigma = \mathrm{sig}(T) \cap \mathrm{sig}(\varphi)$ and set

$$\varphi_n^\Sigma = (|\mathrm{domain}| = n) \rightarrow \bigvee_{M \models T, |M| = n} \psi_{M,\Sigma}$$

where $\psi_{M,S}$ axiomatizes $M_{|\Sigma}$.
$I(T, \varphi) = \{\varphi_n^\Sigma \mid n > 0\}$ is an interpolant:

- $T \models I(T, \varphi)$ is clear;
- $I(T, \varphi) \models \varphi$: if $M \not\models \varphi$, then no expansion of $M_{|\Sigma}$ is a model of T. Hence $M \not\models \varphi_n^\Sigma$ for $n = |M|$.

No interpolation of FO on finite models

- Use $<$ and red to axiomatise in T_1 the theory of finite models M with $|M|$ even; (state that $<$ is a linear order, exactly every second point is red, and the first and last point have distinct colours).
- Use $<'$ and green to axiomatise in T_2 is theory of finite models M with $|M$ odd

Then $T_1 \models_f \neg \bigwedge T_2$, but there does not exist a finite interpolant.

Robustness under joins and interpolation

Theorem

Assume that $\mathcal{QL}$ is closed under Boolean operators and compact. Then the following are equivalent:

- *$\mathcal{QL}$ has interpolation;*
- *$(\mathcal{QL}, \mathcal{QL})$ is robust under joins for possibly infinite sets of sentences.*

We show only "interpolation implies robustness under joins".

Proof

Suppose $\mathcal{QL}$ has interpolation, $T_1 \equiv_\Sigma^{\mathcal{QL}} T_2$ and $\mathrm{sig}(T_1) \cap \mathrm{sig}(T_2) \subseteq \Sigma$.

Let $T_1 \cup T_2 \models \varphi$, where $\mathrm{sig}(\varphi) \subseteq \Sigma$. We show $T_1 \models \varphi$.

- Then (by compactness) for a finite $T_2' \subseteq T_2$

$$T_1 \models \bigwedge T_2' \rightarrow \varphi.$$

- $\bigwedge T_2' \rightarrow \varphi \in \mathcal{QL}$ (closure under Booleans).
- We find interpolant $I(T_1, \bigwedge T_2' \rightarrow \varphi) \in \mathcal{QL}$. Its signature is contained in Σ.
- Then $T_2 \models I(T_1, \bigwedge T_2' \rightarrow \varphi)$, by $T_1 \equiv_\Sigma^{\mathcal{QL}} T_2$.
- Then $T_2 \models \bigwedge T_2' \rightarrow \varphi$. So $T_2 \models \varphi$.

Robustness under joins

The result applies to FO but not to $\mathcal{ALC}$ (nor any other standard DL)! They are NOT closed under Boolean operators! Nevertheless, we can still prove robustness under joins directly:

Theorem

$(\mathcal{QL}, \mathcal{QL})$ is robust under joins for $\mathcal{QL}$ from $\mathcal{ALC}$, $\mathcal{ALCQ}$, $\mathcal{ALCI}$, $\mathcal{ALCQI}$.

Proof in Konev, Lutz, Walther, Wolter: Formal Properties of Modularization, 2008.
On the other hand:

Theorem

$(\mathcal{ALC}_H, \mathcal{ALC}_H)$ and $(\mathcal{ALCO}, \mathcal{ALCO})$ are not robust under joins.

Robustness under Replacement

Theorem

Let $\mathcal{QL}$ be closed under Boolan operators and robust under vocabulary extensions. Then $(\mathcal{QL}, \mathcal{QL})$ is robust under replacement.

Let $T, \varphi \subseteq \mathcal{QL}$ with $\mathrm{sig}(T, \varphi) \cap \mathrm{sig}(T_1 \cup T_2) \subseteq \Sigma$ and

$$T_1 \equiv_{\Sigma}^{\mathcal{QL}} T_2.$$

Then

$$T_1 \models \bigwedge T \rightarrow \varphi \Leftrightarrow T_2 \models \bigwedge T \rightarrow \varphi.$$

Hence

$$T_1 \cup T \models \varphi \Leftrightarrow T_2 \cup T \models \varphi.$$

Robustness under Replacement

Theorem

$(\mathcal{ALC}, \mathcal{ALC})$ *is not robust under replacement.*

Let

$$T_1 = \emptyset, \quad T_2 = \{A \sqsubseteq \exists r.B\}, \quad \Sigma = \{A, B\}.$$

The class of Σ-reducts of models of T_2 is axiomatised by

$$\exists x A(x) \rightarrow \exists x B(x).$$

Hence $T_1 \equiv_{\Sigma}^{\mathcal{ALC}} T_2$. Let $T = \{A \equiv \top, B \equiv \bot\}$. Then

$$T_1 \cup T \not\models \top \sqsubseteq \bot \quad T_2 \cup T \models \top \sqsubseteq \bot.$$

Repairing robustness under replacement

$\mathcal{QL}^B_{\mathcal{ALC}}$ denotes set of Boolean combinations of $\mathcal{ALC}$-concept inclusions.

Theorem

(i) Let T_1 and T_2 be $\mathcal{ALC}$-TBoxes and Σ a signature. Then T_1 and T_2 are Σ-inseparable w.r.t. $\mathcal{QL}^B_{\mathcal{ALC}}$ iff for all models $(\mathcal{I}_1, d_1)$ of T_1 there exists model $(\mathcal{I}_2, d_2)$ of T_2 with

$$(\mathcal{I}_1, d_1) \sim_\Sigma (\mathcal{I}_2, d_2).$$

and the domain and range of $\sim_\Sigma$ coincide with $\Delta^{\mathcal{I}_1}$ and $\Delta^{\mathcal{I}_2}$, respectively, and vice versa.
(ii) $(\mathcal{ALC}, \mathcal{QL}^B_{\mathcal{ALC}})$ is robust under vocabulary extensions, joins, and replacement.
(iii) Σ-inseparability of $\mathcal{ALC}$-TBoxes w.r.t. $\mathcal{QL}^B_{\mathcal{ALC}}$ is 2ExpTime-complete.

See you tomorrow!