

Separating Positive and Negative Data Examples by Concepts and Formulas: The Case of Restricted Signatures

Jean Christoph Jung¹, Carsten Lutz¹, Hadrien Pulcini², and Frank Wolter²

¹University of Bremen, Germany

²University of Liverpool, UK

Abstract. We study the separation of positive and negative data examples in terms of description logic (DL) concepts and formulas of decidable FO fragments, in the presence of an ontology. In contrast to previous work, we add a signature that specifies a subset of the symbols from the data and ontology that can be used for separation. We consider weak and strong versions of the resulting problem that differ in how the negative examples are treated. Our main results are that (a projective form of) the weak version is decidable in \mathcal{ALCI} while it is undecidable in the guarded fragment GF, the guarded negation fragment GNF, and the DL \mathcal{ALCFIO} , and that strong separability is decidable in \mathcal{ALCI} , GF, and GNF. We also provide (mostly tight) complexity bounds.

1 Introduction

There are several applications that fall under the broad term of supervised learning and seek to compute a logical expression that separates positive from negative examples given in the form of labeled data items in a knowledge base. A prominent example is concept learning for description logics (DLs) where the aim is to support a user in automatically constructing a concept description that can then be used, for instance, in ontology engineering [7, 34, 33, 45, 14, 17, 42]. A further example is reverse engineering of database queries (also called query by example, QBE), which has a long history in database research [43, 44, 48, 47, 29, 2, 9, 30, 38] and which has also been studied in the presence of a DL ontology [23, 39]. Note that a closed world semantics is adopted for QBE in databases while an open world semantics is required when the data is assumed to be incomplete as in the presence of ontologies, but also, for example, in reverse engineering of SPARQL queries [3]. Another example is entity comparison in RDF graphs, where one aims to find meaningful descriptions that separate one entity from another [41, 40] and a final example is generating referring expressions (GRE) where the aim is to describe a single data item by a logical expression such as a DL concept, separating it from all other data items. GRE has originated in linguistics [32], but has recently received interest in DL-based ontology-mediated querying [11].

A fundamental problem common to all these applications is to decide whether a separating expression exists at all. There are several degrees of freedom in defining this problem. One concerns the negative examples: is it enough that they

do not entail the separating formula (*weak separability*) or are they required to entail its negation (*strong separability*)? Another one concerns the question whether additional helper symbols are admitted in the separating formula (*projective separability*) or not (*non-projective separability*). The emerging family of problems has recently been investigated in [18, 26], concentrating on the case where the separating expression is a DL concept or formulated in a decidable fragment of first-order logic (FO) such as the guarded fragment (GF) and the guarded negation fragment (GNF).

In this paper, we add a signature Σ that is given as an additional input and require separating expressions to be formulated in Σ (in the non-projective case). This makes it possible to ‘direct’ separation towards expressions based on desired features and to exclude features that are not supposed to be used for separation such as gender and skin color. In the projective case, helper symbols from outside of Σ are also admitted, but must be ‘fresh’ in that they cannot occur in the given knowledge base. Arguably, such fresh symbols make the constructed separating expressions less intuitive from an application perspective and more difficult to understand. However, they sometimes increase the separating power and they emerge naturally from a technical perspective.

The signature Σ brings the separation problem closer to the problem of deciding whether an ontology is a conservative extension of another ontology [25], also a form of separation, and to deciding the existence of uniform interpolants [37]. It turns out, in fact, that lower bounds for these problems can often be adapted to weak separability with signature. In contrast, for strong separability we observe a close connection to Craig interpolation.

We consider both weak and strong separability, generally assuming that the ontology is formulated in the same logic that is used for separation. We concentrate on combined complexity, that is, the input to the decision problems consists of the knowledge base that comprises an ABox and an ontology, the positive and negative examples in the form of lists of individuals (for DLs) or lists of tuples of individuals (for FO fragments that support more than one free variable), and the signature. In the following, we summarize our main results.

We start with weak projective separability in \mathcal{ALCI} , present a characterization in terms of Σ -homomorphisms that generalizes characterizations from [18, 26], and then give a decision procedure based on tree automata. This yields a 2EXPTIME upper bound, and a matching lower bound is obtained by reduction from conservative extensions. In contrast, weak projective (and non-projective) separability in \mathcal{ALCI} without a signature is only NEXPTIME-complete [18]. The non-projective case with signature remains open. We then show that weak separability is undecidable in any fragment of FO that extends GF (such as GNF) or \mathcal{ALCFIO} (such as the two-variable fragment with counting, C^2). In both cases, the proof is by adaptation of undecidability proofs for conservative extensions, from [25] and [19] respectively, and applies to both the projective and the non-projective case. This should be contrasted with the fact that weak separability is decidable and 2EXPTIME-complete for GF and for GNF without a signature, both in the projective and in the non-projective case [26]. The decidability status

of (any version of) separability in \mathcal{ALCFIO} without a signature is open. It is known, however, that projective and non-projective weak separability without a signature are undecidable in the two-variable fragment FO^2 of FO [26].

We then turn to strong separability. Here, the projective and the non-projective case coincide and will thus not be distinguished in what follows. We again start with \mathcal{ALCI} for which we show 2EXPTIME -completeness. The proofs, however, are rather different than in the weak case. For the upper bound, we characterize non-separability in terms of the existence of a set of types that are amalgamable in the sense that they can be realized in a model of the ontology at elements that are all $\mathcal{ALCI}(\Sigma)$ -bisimilar, and that satisfy certain additional properties. To identify sets of amalgamable types, we use an approach that is loosely in the style of type elimination procedures. A matching lower bound is proved by a reduction from the word problem of exponentially space bounded ATMs. We remark that in the strong case, the increase in complexity that results from adding a signature is even more pronounced. In fact, strong separability without a signature is only EXPTIME -complete in \mathcal{ALCI} [26]. We then turn to GF and GNF and establish a close link between strong separability and interpolant existence, the problem to decide for formulas φ, ψ in a language \mathcal{L} whether there exists a formula χ in \mathcal{L} using only the shared symbols of φ and ψ such that both $\varphi \rightarrow \chi$ and $\chi \rightarrow \psi$ are valid. We show that strong separability with signature in GF and GNF are polynomial time reducible to interpolant existence in GF and GNF, respectively. GNF enjoys the Craig interpolation property (CIP), that is, there is such a formula χ whenever $\varphi \rightarrow \psi$ is valid. Thus, from the CIP of GNF and the fact that validity in GNF is 2EXPTIME -complete [8], we obtain that strong separability with signature in GNF is 2EXPTIME -complete. GF fails to have the CIP and a 3EXPTIME upper bound for interpolant existence has only recently been established [28]. We thus obtain a 3EXPTIME upper bound for strong separability with signature in GF that we conjecture to be tight. We note that strong separability without signature is 2EXPTIME -complete in both GNF and GF [26].

2 Preliminaries

Let Σ_{full} be a set of *relation symbols* that contains countably many symbols of every arity $n \geq 1$ and let Const be a countably infinite set of *constants*. A *signature* is a set of relation symbols $\Sigma \subseteq \Sigma_{\text{full}}$. We write \mathbf{a} for a tuple (a_1, \dots, a_n) of constants. A *database* \mathcal{D} is a finite set of *ground atoms* $R(\mathbf{a})$, where $R \in \Sigma_{\text{full}}$ has arity n and \mathbf{a} is a tuple of constants from Const of length n . We use $\text{cons}(\mathcal{D})$ to denote the set of constants that occur in \mathcal{D} .

Denote by FO the set of first-order (FO) formulas constructed from constant-free atomic formulas $x = y$ and $R(\mathbf{x})$, $R \in \Sigma_{\text{full}}$, using conjunction, disjunction, negation, and existential and universal quantification. As usual, we write $\varphi(\mathbf{x})$ to indicate that the free variables in FO-formula φ are all from \mathbf{x} and call a formula *open* if it has at least one free variable and a *sentence* otherwise. Note that we do not admit constants in FO-formulas.

An ontology \mathcal{O} is a finite set of FO-sentences, and a *knowledge base (KB)* is a pair $\mathcal{K} = (\mathcal{O}, \mathcal{D})$ of an ontology \mathcal{O} and a database \mathcal{D} . As usual, KBs $\mathcal{K} = (\mathcal{O}, \mathcal{D})$ are interpreted in *relational structures* $\mathfrak{A} = (\text{dom}(\mathfrak{A}), (R^{\mathfrak{A}})_{R \in \Sigma_{\text{full}}}, (c^{\mathfrak{A}})_{c \in \text{Const}})$ where $\text{dom}(\mathfrak{A})$ is the non-empty *domain* of \mathfrak{A} , each $R^{\mathfrak{A}}$ is a relation over $\text{dom}(\mathfrak{A})$ whose arity matches that of R , and $c^{\mathfrak{A}} \in \text{dom}(\mathfrak{A})$ for all $c \in \text{Const}$. Note that we do not make the *unique name assumption (UNA)*, that is $c_1^{\mathfrak{A}} = c_2^{\mathfrak{A}}$ might hold even when $c_1 \neq c_2$. This is in fact essential for several of our results. A structure \mathfrak{A} is a *model of a KB* $\mathcal{K} = (\mathcal{O}, \mathcal{D})$ if it satisfies all sentences in \mathcal{O} and all ground atoms in \mathcal{D} . A KB \mathcal{K} is *satisfiable* if there exists a model of \mathcal{K} .

We introduce two fragments of FO, the guarded fragment and the description logic \mathcal{ALCI} . In the *guarded fragment (GF)* of FO [1, 21], formulas are built from atomic formulas $R(\mathbf{x})$ and $x = y$ by applying the Boolean connectives and *guarded quantifiers* of the form

$$\forall \mathbf{y}(\alpha(\mathbf{x}, \mathbf{y}) \rightarrow \varphi(\mathbf{x}, \mathbf{y})) \text{ and } \exists \mathbf{y}(\alpha(\mathbf{x}, \mathbf{y}) \wedge \varphi(\mathbf{x}, \mathbf{y}))$$

where $\varphi(\mathbf{x}, \mathbf{y})$ is a guarded formula and $\alpha(\mathbf{x}, \mathbf{y})$ is an atomic formula or an equality $x = y$ that contains all variables in $[\mathbf{x}] \cup [\mathbf{y}]$. The formula α is called the *guard* of the quantifier. We say that an ontology \mathcal{O} is a *GF-ontology* if all formulas in \mathcal{O} are from GF, and likewise for knowledge bases.

We next introduce the DL \mathcal{ALCI} . In this context, unary relation symbols are called *concept names* and binary relation symbols are called *role names* [5, 6]. A *role* is a role name or an *inverse role* R^- with R a role name. For uniformity, we set $(R^-)^- = R$. \mathcal{ALCI} -concepts are defined by the grammar

$$C, D ::= A \mid \neg C \mid C \sqcap D \mid \exists R.C$$

where A ranges over concept names and R over roles. As usual, we write \top to abbreviate $A \sqcup \neg A$ for some fixed concept name A , \top for $\neg \perp$, $C \sqcup D$ for $\neg(\neg C \sqcap \neg D)$, $C \rightarrow D$ for $\neg C \sqcup D$, and $\forall R.C$ for $\neg \exists R.\neg C$. An \mathcal{ALCI} -*concept inclusion (CI)* takes the form $C \sqsubseteq D$ where C and D are \mathcal{ALCI} -concepts. An \mathcal{ALCI} -*ontology* is a finite set of \mathcal{ALCI} -CIs. An \mathcal{ALCI} -*KB* $\mathcal{K} = (\mathcal{O}, \mathcal{D})$ consists of an \mathcal{ALCI} -ontology \mathcal{O} and a database \mathcal{D} . Here and in general in the context of \mathcal{ALCI} , we assume that databases use only unary and binary relation symbols. We sometimes also mention the fragment \mathcal{ALC} of \mathcal{ALCI} in which inverse roles are not available.

To obtain a semantics, every \mathcal{ALCI} -concept C can be translated into an FO-formula C^\dagger with one free variable x :

$$\begin{aligned} A^\dagger &= A(x) \\ (C \sqcap D)^\dagger &= C^\dagger \sqcap D^\dagger \\ (\exists R.C)^\dagger &= \exists y (R(x, y) \wedge C^\dagger[y/x]) \\ (\exists R^-.C)^\dagger &= \exists y (R(y, x) \wedge C^\dagger[y/x]). \end{aligned}$$

The *extension* $C^{\mathfrak{A}}$ of a concept C in a structure \mathfrak{A} is defined as $C^{\mathfrak{A}} = \{a \in \text{dom}(\mathfrak{A}) \mid \mathfrak{A} \models C^\dagger(a)\}$. A CI $C \sqsubseteq D$ is regarded as a shorthand for the FO-sentence $\forall x (C^\dagger(x) \rightarrow D^\dagger(x))$. Thus, every \mathcal{ALCI} -concept can be viewed as

a GF-formula and every \mathcal{ALCI} -ontology can be viewed as a GF-ontology. By economically reusing variables, we can even obtain formulas and ontologies from $\text{GF} \cap \text{FO}^2$. We write $\mathcal{O} \models C \sqsubseteq D$ to say that $\text{CI } C \sqsubseteq D$ is a consequence of ontology \mathcal{O} , that is, $C^{\mathfrak{A}} \subseteq D^{\mathfrak{A}}$ holds in every model \mathfrak{A} of \mathcal{O} . Concepts C and D are *equivalent* w.r.t. an ontology \mathcal{O} if $\mathcal{O} \models C \sqsubseteq D$ and $\mathcal{O} \models D \sqsubseteq C$.

The *Gaifman graph* $G_{\mathfrak{A}}$ of a structure \mathfrak{A} is the undirected graph with set of vertices $\text{dom}(\mathfrak{A})$ and an edge $\{d, e\}$ whenever there exists $\mathbf{a} \in R^{\mathfrak{A}}$ that contains d, e for some relation R . The *distance* $\text{dist}_{\mathfrak{A}}(a, b)$ between $a, b \in \text{dom}(\mathfrak{A})$ is defined as the length of a shortest path from a to b , if such a path exists. Otherwise $\text{dist}_{\mathfrak{A}}(a, b) = \infty$. The *maximal connected component (mcc)* $\mathfrak{A}_{\text{con}(a)}$ of a in \mathfrak{A} is the substructure of \mathfrak{A} induced by the set of all b such that $\text{dist}_{\mathfrak{A}}(a, b) < \infty$.

Let \mathfrak{A} be a structure such that $R^{\mathfrak{A}} = \emptyset$ for any relation symbol R of arity > 2 . We say that \mathfrak{A} is *tree-shaped* if $G_{\mathfrak{A}}$ is a tree without reflexive loops and $R^{\mathfrak{A}} \cap S^{\mathfrak{A}} = \emptyset$ for all distinct roles R, S . We say that \mathfrak{A} has *finite outdegree* if $G_{\mathfrak{A}}$ has finite outdegree. A structure \mathfrak{A} is a *forest structure* w.r.t. an \mathcal{ALCI} -KB $\mathcal{K} = (\mathcal{O}, \mathcal{D})$ if the undirected graph (V, E) with

$$\begin{aligned} V &= \text{dom}(\mathfrak{A}) \\ E &= \{\{d, e\} \mid (d, e) \in R^{\mathfrak{A}} \text{ for some } R\} \setminus \{\{d, e\} \mid d, e \in \text{cons}(\mathcal{D})\} \end{aligned}$$

is a tree. We drop ‘w.r.t. \mathcal{K} ’ if \mathcal{K} is clear from the context and speak of a *forest model* of \mathcal{K} when \mathfrak{A} is a model of \mathcal{K} . The following result is well known.

Lemma 1. *Let \mathcal{K} be an \mathcal{ALCI} -KB and C an \mathcal{ALCI} -concept. If $\mathcal{K} \not\models C(a)$, then there exists a forest model \mathfrak{A} of \mathcal{K} of finite outdegree with $a \notin C^{\mathfrak{A}}$.*

We close this section with introducing homomorphisms and bisimulations. Let Σ be a signature. A Σ -*homomorphism* h from a structure \mathfrak{A} to a structure \mathfrak{B} is a function $h : \text{dom}(\mathfrak{A}) \rightarrow \text{dom}(\mathfrak{B})$ such that $\mathbf{a} \in R^{\mathfrak{A}}$ implies $h(\mathbf{a}) \in R^{\mathfrak{B}}$ for all relation symbols $R \in \Sigma$ and tuples \mathbf{a} and with $h(\mathbf{a})$ being defined component wise in the expected way. Note that homomorphisms need not preserve constant symbols. Every database \mathcal{D} gives rise to the finite structure $\mathfrak{A}_{\mathcal{D}}$ with $\text{dom}(\mathfrak{A}_{\mathcal{D}}) = \text{cons}(\mathcal{D})$ and $\mathbf{a} \in R^{\mathfrak{A}_{\mathcal{D}}}$ iff $R(\mathbf{a}) \in \mathcal{D}$. A Σ -homomorphism from database \mathcal{D} to structure \mathfrak{A} is a Σ -homomorphism from $\mathfrak{A}_{\mathcal{D}}$ to \mathfrak{A} . A *pointed structure* takes the form \mathfrak{A}, \mathbf{a} with \mathfrak{A} a structure and \mathbf{a} a tuple of elements of $\text{dom}(\mathfrak{A})$. A homomorphism from pointed structure \mathfrak{A}, \mathbf{a} to pointed structure \mathfrak{B}, \mathbf{b} is a homomorphism h from \mathfrak{A} to \mathfrak{B} with $h(\mathbf{a}) = \mathbf{b}$. We write $\mathfrak{A}, \mathbf{a} \rightarrow \mathfrak{B}, \mathbf{b}$ to indicate the existence of such a homomorphism.

We introduce $\mathcal{ALCI}(\Sigma)$ -bisimulations between structures \mathfrak{A} and \mathfrak{B} that interpret relations of arity at most two. Let Σ be a signature. A relation $S \subseteq \text{dom}(\mathfrak{A}) \times \text{dom}(\mathfrak{B})$ is an $\mathcal{ALCI}(\Sigma)$ -*bisimulation between \mathfrak{A} and \mathfrak{B}* if the following conditions hold:

1. for all $(d, e) \in S$: $d \in A^{\mathfrak{A}}$ iff $e \in A^{\mathfrak{B}}$;
2. if $(d, e) \in S$ and $(d, d') \in R^{\mathfrak{A}}$, then there is a e' with $(e, e') \in R^{\mathfrak{B}}$ and $(d', e') \in S$;
3. if $(d, e) \in S$ and $(e, e') \in R^{\mathfrak{B}}$, then there is a d' with $(d, d') \in R^{\mathfrak{A}}$ and $(d', e') \in S$,

where A ranges over all concept names in Σ and R over all Σ -roles. We write $\mathfrak{A}, d \sim_{\mathcal{ALCI}, \Sigma} \mathfrak{B}, e$ and call \mathfrak{A}, d and \mathfrak{B}, e $\mathcal{ALCI}(\Sigma)$ -bisimilar if there exists an $\mathcal{ALCI}(\Sigma)$ -bisimulation S such that $(d, e) \in S$.

The next lemma explains why $\mathcal{ALCI}(\Sigma)$ -bisimulations are relevant [35, 20]. We say that \mathfrak{A}, d and \mathfrak{B}, e are $\mathcal{ALCI}(\Sigma)$ -equivalent, in symbols $\mathfrak{A}, d \equiv_{\mathcal{ALCI}, \Sigma} \mathfrak{B}, e$ if $d \in C^{\mathfrak{A}}$ iff $e \in C^{\mathfrak{B}}$ for all $C \in \mathcal{ALCI}(\Sigma)$.

Lemma 2. *Let \mathfrak{A}, d and \mathfrak{B}, e be pointed structures of finite outdegree and Σ a signature. Then*

$$\mathfrak{A}, d \equiv_{\mathcal{ALCI}, \Sigma} \mathfrak{B}, e \text{ iff } \mathfrak{A}, d \sim_{\mathcal{ALCI}, \Sigma} \mathfrak{B}, e.$$

For the “if”-direction, the condition on the outdegree can be dropped.

For any syntactic object O such as a formula, an ontology, and a KB, we use $\text{sig}(O)$ to denote the set of relation symbols that occur in O and $\|O\|$ to denote the size of O , that is, the number of symbols needed to write it with names of relations, variables, and constants counting as a single symbol.

3 Weak Separability With Signature

We start with introducing the problem of (weak) separability with signature, in its projective and non-projective version.

Definition 1. *Let \mathcal{L} be a fragment of FO. A labeled \mathcal{L} -KB takes the form (\mathcal{K}, P, N) with $\mathcal{K} = (\mathcal{O}, \mathcal{D})$ an \mathcal{L} -KB and $P, N \subseteq \text{cons}(\mathcal{D})^n$ non-empty sets of positive and negative examples, all of them tuples of the same length n .*

Let $\Sigma \subseteq \text{sig}(\mathcal{K})$ be a signature. An FO(Σ)-formula $\varphi(\mathbf{x})$ with n free variables Σ -separates (\mathcal{K}, P, N) if $\text{sig}(\mathcal{K}) \cap \text{sig}(\varphi) \subseteq \Sigma$ and

1. $\mathcal{K} \models \varphi(\mathbf{a})$ for all $\mathbf{a} \in P$ and
2. $\mathcal{K} \not\models \varphi(\mathbf{a})$ for all $\mathbf{a} \in N$.

Let \mathcal{L}_S be a fragment of FO. We say that (\mathcal{K}, P, N) is projectively $\mathcal{L}_S(\Sigma)$ -separable if there is an \mathcal{L}_S -formula $\varphi(\mathbf{x})$ that Σ -separates (\mathcal{K}, P, N) and (non-projectively) $\mathcal{L}_S(\Sigma)$ -separable if there is such a $\varphi(\mathbf{x})$ with $\text{sig}(\varphi) \subseteq \Sigma$.

Relation symbols in Σ -separating formulas that are not from Σ should be thought of as helper symbols. Their availability sometimes makes inseparable KBs separable, examples are provided below. We only consider FO-fragments \mathcal{L}_S that are closed under conjunction. In this case, a labeled KB (\mathcal{K}, P, N) is $\mathcal{L}_S(\Sigma)$ -separable if and only if all $(\mathcal{K}, P, \{\mathbf{b}\})$, $\mathbf{b} \in N$, are $\mathcal{L}_S(\Sigma)$ -separable, and likewise for projective $\mathcal{L}_S(\Sigma)$ -separability, see [26]. In what follows, we thus mostly consider labeled KBs with singleton sets N of negative examples.

Each choice of an ontology language \mathcal{L} and a separation language \mathcal{L}_S give rise to a projective and to a non-projective separability problem. In the current paper, we only consider cases where $\mathcal{L} = \mathcal{L}_S$ (see below for a discussion).

PROBLEM : (Projective) \mathcal{L} -separability with signature
INPUT : A labeled \mathcal{L} -KB (\mathcal{K}, P, N) and signature $\Sigma \subseteq \text{sig}(\mathcal{K})$
QUESTION : Is (\mathcal{K}, P, N) (projectively) $\mathcal{L}(\Sigma)$ -separable?

We study the *combined complexity* of \mathcal{L} -separability with signature where the ontology \mathcal{O} , database \mathcal{D} (both in \mathcal{K}), and sets of examples P and N are all taken to be part of the input. One can also study *data complexity* where only \mathcal{D} , P , and N are regarded as inputs while \mathcal{O} is assumed to be fixed [26]. A special case of \mathcal{L} -separability with signature is \mathcal{L} -*definability* with signature where P and N partition the example space, that is, inputs are labeled \mathcal{L} -KBs (\mathcal{K}, P, N) such that $N = \text{cons}(\mathcal{D})^n \setminus P$, n the length of example tuples. All our results also hold for definability.

We now give some examples illustrating the central notions used in this paper. In [26], projective and non-projective separability are studied without signature restrictions. Thus, all symbols used in the KB can appear in separating formulas. Rather surprisingly, it turned out that in this case many different separation languages have exactly the same separating power. For example, a labeled FO-KB turned out to be FO-separable iff it is UCQ-separable¹ (and projective and non-projective separability coincide) and a labeled \mathcal{ALCC} -KB is projectively \mathcal{ALCC} -separable iff it is (non-)projectively FO-separable. No such result can be expected for separability with signature restrictions, as illustrated by the following example.

Example 1. Let $\mathcal{O} = \{A \sqsubseteq \exists R.B \sqcap \exists R.\neg B\}$ and $\mathcal{D} = \{A(a), R(b, c)\}$. Let $P = \{a\}$, $N = \{b\}$, and $\Sigma = \{R\}$. Clearly, the formula

$$\exists y \exists y' (R(x, y) \wedge R(x, y') \wedge \neg(y = y'))$$

Σ -separates $(\mathcal{O}, \mathcal{D}, P, N)$, but $(\mathcal{O}, \mathcal{D}, P, N)$ is neither UCQ(Σ)-separable nor $\mathcal{ALCC}(\Sigma)$ -separable.

However, the ability to restrict separating formulas to a given signature makes it possible to guide separation towards desired aspects.

Example 2. Consider a KB about books that uses, say, the concept and role names provided by schema.org (called types and properties there). Schema.org offers dozens of such names related to books, ranging from editor, author, and illustrator to genre, date published, and character. Assume that a few books have been labeled as *likes* (added to P) and *dislikes* (added to N) and one would like to find a formula φ that separate P from N . Then it might be useful to restrict the signature of φ so as to concentrate on the aspects of books that one is most interested in. For example, one could select a signature that contains symbols related to genre such as graphic novel, adventure, classic, and drop all remaining symbols. If no separating formula exists, one can then iteratively

¹ We denote by UCQ the set of FO-formulas that are disjunctions of formulas constructed from atoms using conjunction and existential quantification.

extend the signature until separating formulas are found. We refer the reader to research on modules and modularity in ontologies, where signatures are also used to capture the topic of a module [22, 31, 12, 13]. If one is not sure which aspects are most relevant for separation, one might of course also decide to work with a large signature. But also in such a case, it might be useful to exclude certain undersired symbols such as the author’s age and gender.

The helper symbols that distinguish the projective from the non-projective case play a completely different role from the symbols in the signature Σ selected for separation, as discussed next.

Example 3. Consider a database \mathcal{D} in which an individual a is part of an R -cycle and b has both an R -reflexive successor and predecessor. Thus,

$$\mathcal{D} = \{R(a_0, a_1), \dots, R(a_{n-1}, a_n), R(b, b_1), R(b_1, b_1), R(b_2, b), R(b_2, b_2)\},$$

where $a_0 = a_n = a$ and $n > 0$. Let \mathcal{O} be either empty or any \mathcal{ALCI} -ontology such that $\mathcal{K} = (\mathcal{O}, \mathcal{D})$ is satisfiable and $\top \sqsubseteq \exists R.\top \sqcap \exists R^{\neg}.\top \in \mathcal{O}$. Further let $P = \{a\}$, $N = \{b\}$, and $\Sigma = \{R\}$. If no helper symbols are allowed, then (\mathcal{K}, P, N) is not $\mathcal{ALCI}(\Sigma)$ -separable (we show this in Example 4 below). If, however, a helper symbol A is allowed, then $\neg A \sqcup \exists R^n.A$ is a separating \mathcal{ALCI} -concept. Thus, (\mathcal{K}, P, N) is projectively $\mathcal{ALCI}(\Sigma)$ -separable but not non-projectively $\mathcal{ALCI}(\Sigma)$ -separable. The use of a cycle in this example is no accident; we show in the next section that if there are no cycles in the database, then helper symbols do not add any separating power to \mathcal{ALCI} -concepts.

4 Weak Separability in \mathcal{ALCI}

We give a model-theoretic characterization of projective weak \mathcal{ALCI} -separability with signature and use it to prove decidability in 2EXPTIME . A matching lower bound is obtained by reduction from conservative extensions. We also use the characterization to discuss the relationship between non-projective and projective separability, showing, in particular, that on tree-shaped databases the two notions coincide.

We start with a model-theoretic characterization of projective $\mathcal{ALCI}(\Sigma)$ -separability of labeled KBs. As \mathcal{ALCI} -concepts talk about individual elements and not tuples, we assume in this section that examples in labeled KBs are constants from the database. In fact, we give three characterizations that are more and more refined. We use the second one to clarify the relationship between projective and non-projective separability and the third characterization for the decision procedure. The first characterization directly reflects that we are considering projective separability with signature as it is based on $\mathcal{ALCI}(\Sigma')$ -bisimulations where Σ' is a signature satisfying $\Sigma' \cap \text{sig}(\mathcal{K}) \subseteq \Sigma$. The second characterization replaces $\mathcal{ALCI}(\Sigma')$ -bisimulations by functional $\mathcal{ALCI}(\Sigma)$ -bisimulations. The final characterization replaces the functional bisimulations from the second characterization by a combination of Σ -homomorphisms and $\mathcal{ALCI}(\Sigma)$ -bisimulations. We first introduce some required notation.

An *extended database* is a database that additionally may contain ‘atoms’ of the form $C(a)$ with C an \mathcal{ALCI} -concept. The semantics of extended databases is defined in the expected way. We write $\mathfrak{A}, a \sim_{\mathcal{ALCI}, \Sigma}^f \mathfrak{B}, b$ if there exists an $\mathcal{ALCI}(\Sigma)$ -bisimulation S between \mathfrak{A} and \mathfrak{B} that contains (a, b) and is functional, that is, $(d, d_1), (d, d_2) \in S$ imply $d_1 = d_2$. Let $\text{sub}(\mathcal{K})$ denote the set of concepts that occur in \mathcal{K} , closed under single negation and under subconcepts. The \mathcal{K} -type realized in a pointed structure \mathfrak{A}, a is defined as

$$\text{tp}_{\mathcal{K}}(\mathfrak{A}, a) = \{C \in \text{sub}(\mathcal{K}) \mid a \in C^{\mathfrak{A}}\}.$$

A \mathcal{K} -type is any set $t \subseteq \text{sub}(\mathcal{K})$ of the form $\text{tp}_{\mathcal{K}}(\mathfrak{A}, a)$. For a pointed database \mathcal{D}, a , we write $\mathcal{D}_{\text{con}(a)}, a \rightarrow_c^{\Sigma} \mathfrak{A}, b$ if there is a Σ -homomorphism h from the maximal connected component $\mathcal{D}_{\text{con}(a)}$ of a in \mathcal{D} to \mathfrak{A} such that $h(a) = b$ and there is a \mathcal{K} -type t_d for each $d \in \text{cons}(\mathcal{D}_{\text{con}(a)})$ such that:

1. there exists a model \mathfrak{B}_d of \mathcal{O} with $\text{tp}_{\mathcal{K}}(\mathfrak{B}_d, d) = t_d$ and $\mathfrak{B}_d, d \sim_{\mathcal{ALCI}, \Sigma} \mathfrak{A}, h(d)$;
2. $(\mathcal{O}, \mathcal{D}')$ is satisfiable, for the extended database $\mathcal{D}' = \mathcal{D} \cup \{C(d) \mid C \in t_d, d \in \text{cons}(\mathcal{D}_{\text{con}(a)})\}$.

Theorem 1. *Let $(\mathcal{K}, P, \{b\})$ be a labeled \mathcal{ALCI} -KB with $\mathcal{K} = (\mathcal{O}, \mathcal{D})$ and $\Sigma \subseteq \text{sig}(\mathcal{K})$. Then the following conditions are equivalent:*

1. $(\mathcal{K}, P, \{b\})$ is projectively $\mathcal{ALCI}(\Sigma)$ -separable.
2. there exists a forest model \mathfrak{A} of \mathcal{K} of finite outdegree and a signature Σ' such that $\Sigma' \cap \text{sig}(\mathcal{K}) \subseteq \Sigma$ and for all models \mathfrak{B} of \mathcal{K} and all $a \in P$: $\mathfrak{B}, a^{\mathfrak{B}} \not\sim_{\mathcal{ALCI}, \Sigma'} \mathfrak{A}, b^{\mathfrak{A}}$.
3. there exists a forest model \mathfrak{A} of \mathcal{K} of finite outdegree such that for all models \mathfrak{B} of \mathcal{K} and all $a \in P$: $\mathfrak{B}, a^{\mathfrak{B}} \not\sim_{\mathcal{ALCI}, \Sigma}^f \mathfrak{A}, b^{\mathfrak{A}}$.
4. there exists a forest model \mathfrak{A} of \mathcal{K} of finite outdegree such that for all $a \in P$: $\mathcal{D}_{\text{con}(a)}, a \not\rightarrow_c^{\Sigma} \mathfrak{A}, b^{\mathfrak{A}}$.

The proof relies on Lemmas 1 and 2 and is given in the appendix.

We first use Theorem 1 to discuss the relationship between projective and non-projective separability. A basic model-theoretic characterization of non-projective $\mathcal{ALCI}(\Sigma)$ -separability is rather straightforward to obtain by just dropping the quantification over Σ' in Condition 2 of Theorem 1 and demanding instead that $\mathfrak{B}, a^{\mathfrak{B}} \not\sim_{\mathcal{ALCI}, \Sigma} \mathfrak{A}, b^{\mathfrak{A}}$, for all models \mathfrak{B} of \mathcal{K} and $a \in P$. As a consequence, one can then also adapt Condition 3 of Theorem 1 to the non-projective case by simply dropping the functionality condition on the bisimulation. If the database $\mathcal{D}_{\text{con}(a)}$ is tree-shaped, then there is no difference between the two versions of Condition 2 as one can always introduce sufficiently many copies of nodes in models \mathfrak{B} of \mathcal{K} to turn an unrestricted bisimulation into a functional one. We obtain the following result.

Theorem 2. *Let (\mathcal{K}, P, N) be a labeled \mathcal{ALCI} -KB with $\mathcal{K} = (\mathcal{O}, \mathcal{D})$ such that $\mathcal{D}_{\text{con}(a)}$ is tree-shaped for all $a \in P$ and $\Sigma \subseteq \text{sig}(\mathcal{K})$. Then (\mathcal{K}, P, N) is projectively $\mathcal{ALCI}(\Sigma)$ -separable iff it is non-projectively $\mathcal{ALCI}(\Sigma)$ -separable.*

The following example illustrates Theorem 1.

Example 4. Consider the labeled KB (\mathcal{K}, P, N) and signature Σ from Example 3. Then we find a model \mathfrak{A} of \mathcal{K} of finite outdegree such that $b^{\mathfrak{A}}$ does not participate in any R -cycle. Then, as a participates in an R -cycle, there does not exist any model \mathfrak{B} of \mathcal{K} such that $\mathfrak{B}, a^{\mathfrak{B}} \sim_{\mathcal{ALCI}, \Sigma}^f \mathfrak{A}, b^{\mathfrak{A}}$, and so (\mathcal{K}, P, N) is projectively $\mathcal{ALCI}(\Sigma)$ -separable. (\mathcal{K}, P, N) is not non-projectively $\mathcal{ALCI}(\Sigma)$ -separable: if \mathcal{O} contains $\top \sqsubseteq \exists R.\top \sqcap \exists R^-\top$, then $S = \text{dom}(\mathfrak{B}) \times \text{dom}(\mathfrak{A})$ is an $\mathcal{ALCI}(\Sigma)$ -bisimulation between \mathfrak{B} and \mathfrak{A} , for any model \mathfrak{B} of \mathcal{K} . If \mathcal{O} is empty, then the construction of the required bisimulation is also straightforward.

We now come to the main result of this section.

Theorem 3. *Projective \mathcal{ALCI} -separability with signature is 2EXPTIME-complete.*

The upper bound in Theorem 3 is obtained by using the characterization in Condition 4 of Theorem 1 to devise a decision procedure based on tree automata. Given a labeled \mathcal{ALCI} -KB $(\mathcal{K}, P, \{b\})$, we construct a tree automaton \mathcal{A} such that the language recognized by \mathcal{A} is non-empty if and only if there is a forest model \mathfrak{A} of \mathcal{K} as described in Condition 4 of Theorem 1. We use a variant of two-way alternating parity tree automata over infinite trees [46]. In contrast to the standard model, our automata work on trees of finite, but not necessarily bounded outdegree. Such an automata model has been introduced in [25]. We recall the technical preliminaries.

A *tree* is a non-empty (and potentially infinite) set of words $T \subseteq (\mathbb{N} \setminus 0)^*$ closed under prefixes. We generally assume that trees are finitely branching, that is, for every $w \in T$, the set $\{i \mid w \cdot i \in T\}$ is finite. For any $w \in (\mathbb{N} \setminus 0)^*$, as a convention we set $w \cdot 0 := w$. If $w = n_0 n_1 \cdots n_k$, we additionally set $w \cdot -1 := n_0 \cdots n_{k-1}$. For an alphabet Θ , a Θ -labeled tree is a pair (T, L) with T a tree and $L : T \rightarrow \Theta$ a node labeling function.

A *two-way alternating tree automaton (2ATA)* is a tuple $\mathcal{A} = (Q, \Theta, q_0, \delta, \Omega)$ where Q is a finite set of *states*, Θ is the finite *input alphabet*, $q_0 \in Q$ is the *initial state*, δ is a *transition function* as specified below, and $\Omega : Q \rightarrow \mathbb{N}$ is a *priority function*. The automaton runs on Θ -labeled trees. The transition function maps a state q and some input letter $\theta \in \Theta$ to a *transition condition* $\delta(q, \theta)$ which is a positive Boolean formula over the truth constants **true** and **false** and transitions of the form $q, \langle - \rangle q, [-]q, \diamond q, \square q$ where $q \in Q$. Informally, the transition q expresses that a copy of the automaton is sent to the current node in state q , $\langle - \rangle q$ means that a copy is sent in state q to the predecessor node, which is then required to exist, $[-]q$ means the same except that the predecessor node is not required to exist, $\diamond q$ means that a copy is sent in state q to some successor, and $\square q$ that a copy is sent in state q to all successors. The semantics is defined in terms of runs in the usual way, we refer to [25] for details. We use $L(\mathcal{A})$ to denote the set of all Θ -labeled trees accepted by \mathcal{A} . 2ATAs are closed under complementation and intersection, and their emptiness problem, which asks whether $L(\mathcal{A}) = \emptyset$ for a given 2ATA \mathcal{A} , can be decided in time exponential in the number of states of \mathcal{A} [25].

The input alphabet Θ consists of two types of symbols:

1. models \mathfrak{A}_0 of \mathcal{D} with $\text{dom}(\mathfrak{A}_0) \subseteq D_0$, D_0 a fixed set of cardinality $|\text{cons}(\mathcal{D})|$;
2. triples (a, R, M) for $a \in \text{cons}(\mathcal{D})$, R a role used in \mathcal{O} , and $M \subseteq \text{sig}(\mathcal{O}) \cap \mathbf{Nc}$.

A Θ -labeled tree is *well-formed* if it has a label of Type 1 at the root and labels of Type 2 everywhere else. A well-formed Θ -labeled tree τ encodes a structure \mathfrak{A}_τ that can be constructed as follows:

- start with $\mathfrak{A}_\tau = \mathfrak{A}_0$, the structure from the root label;
- for every non-root $v \in T$ with $\tau(v) = (a, R, M)$, extend \mathfrak{A}_τ as follows:
 - if the predecessor of v is the root, add an R -successor a_v of $a^{\mathfrak{A}_\tau}$;
 - if the predecessor v' of v is not the root, add an R -successor a_v of $a_{v'}$ (so a is ignored in this case and should be considered a dummy.)

In both cases, a_v makes true exactly the concept names in M .

It should be clear that every structure \mathfrak{A}_τ is a forest structure. Conversely, we can encode every forest structure \mathfrak{A} into a Θ -labeled tree τ such that $\mathfrak{A}_\tau = \mathfrak{A}$.

Lemma 3. *There are 2ATAs $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ such that:*

1. \mathcal{A}_1 accepts precisely the well-formed Θ -labeled trees;
2. \mathcal{A}_2 accepts a well-formed Θ -labeled tree τ iff \mathfrak{A}_τ is a model of \mathcal{K} ;
3. \mathcal{A}_3 accepts a well-formed Θ -labeled tree τ if $\mathcal{D}_{\text{con}(a), a \rightarrow_c^\Sigma \mathfrak{A}_\tau, b^{\mathfrak{A}_\tau}}$.

The number of states of \mathcal{A}_1 is 2; the number of states of \mathcal{A}_2 is polynomial in $\|\mathcal{O}\|$; the number of states of \mathcal{A}_3 is exponential in $\|\mathcal{K}\|$. Moreover, $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ can be constructed in time double exponential in $\|\mathcal{K}\|$.

As the construction of \mathcal{A}_1 and \mathcal{A}_2 is rather standard, we only sketch the construction of the automaton \mathcal{A}_3 . See e.g. [25] for full details of a similar construction.

As the first step, \mathcal{A}_3 reads the symbol \mathfrak{A}_0 at the root and non-deterministically guesses the following:

- types t_d , $d \in \text{cons}(\mathcal{D}_{\text{con}(a)})$, such that $(\mathcal{O}, \mathcal{D}')$ is satisfiable where $\mathcal{D}' = \mathcal{D} \cup \{C(d) \mid C \in t_d, d \in \text{cons}(\mathcal{D}_{\text{con}(a)})\}$;
- a partition $\mathcal{D}_0, \mathcal{D}_1, \dots, \mathcal{D}_m$ of $\mathcal{D}_{\text{con}(a)}$ such that $\text{cons}(\mathcal{D}_i) \cap \text{cons}(\mathcal{D}_j) = \emptyset$ for $1 \leq i < j \leq m$;
- a Σ -homomorphism h from \mathcal{D}_0 to \mathfrak{A}_0 such that $h(a) = b^{\mathfrak{A}_0}$ and there are a_1, \dots, a_m with $h(c) = a_i$ for all $c \in \text{cons}(\mathcal{D}_0 \cap \mathcal{D}_i)$ and $1 \leq i \leq m$.

The entire guess is stored in the state of the automaton. Note that the first item checks that Point 2 from the definition of $\mathcal{D}_{\text{con}(a), a \rightarrow_c^\Sigma \mathfrak{A}_\tau, b^{\mathfrak{A}_\tau}}$ is satisfied for the guessed types t_d . After making its guess, the automaton verifies that the homomorphism h from the last item can be extended to a homomorphism from $\mathcal{D}_{\text{con}(a)}$ to \mathfrak{A}_τ that satisfies Point 1 from the definition of $\mathcal{D}_{\text{con}(a), a \rightarrow_c^\Sigma \mathfrak{A}_\tau, b^{\mathfrak{A}_\tau}}$. To this end, it does a top-down traversal of \mathfrak{A}_τ checking that each \mathcal{D}_i can be homomorphically mapped to the subtree of \mathfrak{A}_τ below $h(a_i)$. During the traversal, the automaton memorizes in its state the set of constants from \mathcal{D}_i that are mapped to the currently visited element.

The automaton additionally makes sure that Point 1 from the definition of $\mathcal{D}_{\text{con}(a), a \rightarrow_c^\Sigma \mathfrak{A}_\tau, b^{\mathfrak{A}_\tau}}$ is satisfied, in the following way. During the top-down

traversal, it spawns copies of itself to verify that, whenever it has decided to map a $d \in \text{cons}(\mathcal{D}_{\text{con}(a)})$ to the current element, then there is a tree-shaped model \mathcal{B}_d of \mathcal{O} with $\text{tp}_{\mathcal{K}}(\mathcal{B}_d, d) = t_d$ and a bisimulation that witnesses $\mathcal{B}_d, d \sim_{\mathcal{ALCT}, \Sigma} \mathfrak{A}_\tau, c$. This is done by ‘virtually’ traversing \mathcal{B}_d elements-by-element, storing at each moment only the type of the current element in a state. This is possible because \mathcal{B}_d is tree-shaped. At the beginning, the automaton is at an element of \mathcal{B}_d of type t_d and knows that the bisimulation maps this element to the node of \mathfrak{A}_τ currently visited by the automaton. It then does two things to verify the two main conditions of bisimulations. First, it transitions to every neighbor of the node of \mathfrak{A}_τ currently visited, both upwards and downwards, and carries out in its state the corresponding transition in \mathcal{B}_d , in effect guessing a new type. Second, it considers the current type of \mathcal{B}_d and guesses successor types that satisfy the existential restrictions in it. For every required successor type, it then guesses a neighbor of the currently visited node in \mathfrak{A}_τ to which the successor is mapped. The two steps are alternated, exploiting the alternation capabilities of the automaton. Some extra bookkeeping in states is needed for the root node of the input tree as it represents more than one element of \mathfrak{A}_τ .

It can be verified that only exponentially many states are required and that the transition function can be computed in double exponential time. This finishes the proof sketch of Lemma 3.

The upper bound in Theorem 3 is now obtained as follows. By Condition 4 of Theorem 1 and Lemma 3, $(\mathcal{K}, P, \{b\})$ is projectively $\mathcal{ALCT}(\Sigma)$ -separable iff $L(\mathcal{A}_1) \cap L(\mathcal{A}_2) \cap \overline{L(\mathcal{A}_3)}$ is not empty where $\overline{L(\mathcal{A}_3)}$ denotes the complement of $L(\mathcal{A}_3)$. By Lemma 3, all \mathcal{A}_i can be constructed in double exponential time and their number of states is single exponential in $\|\mathcal{K}\|$. As the complement and intersections of 2ATAs can be computed in polynomial time with only a polynomial increase in the number of states, it remains to recall that non-emptiness of 2ATAs can be decided in time exponential in the number of states.

For the lower bound, we reduce from conservative extensions in \mathcal{ALCT} . An \mathcal{ALCT} -ontology \mathcal{O}_2 is a *conservative extension* of an \mathcal{ALCT} -ontology $\mathcal{O}_1 \subseteq \mathcal{O}_2$ if there is no $\mathcal{ALCT}(\text{sig}(\mathcal{O}_1))$ -concept C that is satisfiable w.r.t. \mathcal{O}_1 , but unsatisfiable w.r.t. \mathcal{O}_2 . We define *projective conservative extensions* in the same way except that C is now an \mathcal{ALCT} -concept with $\text{sig}(\mathcal{O}_2) \cap \text{sig}(C) \subseteq \text{sig}(\mathcal{O}_1)$. It was shown in [19] that it is 2EXPTIME-hard to decide, given \mathcal{ALCT} -ontologies \mathcal{O}_1 and \mathcal{O}_2 , whether \mathcal{O}_2 is a (non-projective) conservative extensions of \mathcal{O}_1 . It was further observed that conservative extensions and projective conservative extensions coincide in logics that enjoy Craig interpolation, which \mathcal{ALCT} does [25]. Thus, projective conservative extensions in \mathcal{ALCT} are also 2EXPTIME-hard. We give a polynomial time reduction from (the complement of) that problem to projective \mathcal{ALCT} -separability with signature.

Thus let $\mathcal{O}_1, \mathcal{O}_2$ be \mathcal{ALCT} -ontologies with $\mathcal{O}_1 \subseteq \mathcal{O}_2$. We can assume w.l.o.g. that \mathcal{O}_1 takes the form $\{\top \sqsubseteq C_1\}$, \mathcal{O}_2 takes the form $\mathcal{O}_1 \cup \{\top \sqsubseteq C_2\}$, and that \mathcal{O}_2 is satisfiable. For a concept name A , the *A-relativization* C^A of an \mathcal{ALCT} -concept C is obtained by replacing every subconcept $\exists r.D$ in C with $\exists r.(A \sqcap D)$.

Define

$$\mathcal{O} = \{\top \sqsubseteq C_1^A, A \sqsubseteq C_2^A\}$$

where A is a concept name that does not occur in \mathcal{O}_2 . Then \mathcal{O}_2 is not a projective conservative extension of \mathcal{O}_1 iff there is an $\mathcal{ALCI}(\text{sig}(\mathcal{O}_1))$ -concept C that is satisfiable w.r.t. \mathcal{O}_1 but unsatisfiable w.r.t. \mathcal{O}_2 iff the labeled \mathcal{ALCI} -KB $(\mathcal{K}, \{a\}, \{b\})$ is projectively $\mathcal{ALCI}(\text{sig}(\mathcal{O}_1))$ -separable by $\neg C$ where $\mathcal{K} = (\mathcal{O}, \mathcal{D})$ and $\mathcal{D} = \{A(a), D(b)\}$, D a fresh (dummy) concept name. We have thus established the lower bound from Theorem 3.

We leave open the decidability and exact complexity of non-projective \mathcal{ALCI} -separability with signature. A 2EXPTIME lower bound can be established along the lines above.

5 Weak Separability: Undecidable Cases

Inspired by the close connection of conservative extensions and weak separability with signature that was established in the previous section, we investigate logics for which conservative extensions are undecidable: the guarded fragment GF and the expressive DL \mathcal{ALCFIO} . The latter logic, \mathcal{ALCFIO} , is the extension of \mathcal{ALCI} with *nominals* and *functionality assertions*. Nominals are concepts of the form $\{o\}$ with o a constant symbol, and the translation \cdot^\dagger from \mathcal{ALCI} into FO can be extended to nominals by setting $\{o\}^\dagger = (x = o)$. Functionality assertions are concept inclusions of the form $\top \sqsubseteq (\leq 1 r)$, r a role, which demand that the role r is interpreted as a partial function. Thus, they are a weak form of counting, and indeed \mathcal{ALCFIO} is a fragment of C^2 , the two-variable fragment of FO with counting.

It is known that conservative extensions and projective conservative extensions are undecidable in every extension of the three-variable fragment GF^3 of GF [25] and in every extension of \mathcal{ALCFIO} [36]. Unfortunately, it is not clear how to achieve a direct reduction of conservative extensions to separability for both GF and \mathcal{ALCFIO} . In both cases, the relativization that was used for \mathcal{ALCI} does not work. For GF, this is the case because non-conservativity in GF is witnessed by *sentences* while separability is witnessed by *formulas*. For \mathcal{ALCFIO} , relativization cannot be applied due to the presence of nominals/constants. We instead directly use and adapt the strategies of the mentioned undecidability proofs, starting with GF.

Theorem 4. *Projective and non-projective \mathcal{L} -separability with signature are undecidable for every logic \mathcal{L} that contains GF^3 . This is even true when the language of the separating formula is \mathcal{ALC} .*

The proof is by a reduction from the halting problem of two-register machines. A (deterministic) *two-register machine (2RM)* is a pair $M = (Q, P)$ with $Q = q_0, \dots, q_\ell$ a set of *states* and $P = I_0, \dots, I_{\ell-1}$ a sequence of *instructions*. By definition, q_0 is the *initial state*, and q_ℓ the *halting state*. For all $i < \ell$,

- either $I_i = +(p, q_j)$ is an *incrementation instruction* with $p \in \{0, 1\}$ a register and q_j the subsequent state;
- or $I_i = -(p, q_j, q_k)$ is a *decrementation instruction* with $p \in \{0, 1\}$ a register, q_j the subsequent state if register p contains 0, and q_k the subsequent state otherwise.

A *configuration* of M is a triple (q, m, n) , with q the current state and $m, n \in \mathbb{N}$ the register contents. We write $(q_i, n_1, n_2) \Rightarrow_M (q_j, m_1, m_2)$ if one of the following holds:

- $I_i = +(p, q_j)$, $m_p = n_p + 1$, and $m_{1-p} = n_{1-p}$;
- $I_i = -(p, q_j, q_k)$, $n_p = m_p = 0$, and $m_{1-p} = n_{1-p}$;
- $I_i = -(p, q_k, q_j)$, $n_p > 0$, $m_p = n_p - 1$, and $m_{1-p} = n_{1-p}$.

The *computation* of M on input $(n, m) \in \mathbb{N}^2$ is the unique longest configuration sequence $(p_0, n_0, m_0) \Rightarrow_M (p_1, n_1, m_1) \Rightarrow_M \dots$ such that $p_0 = q_0$, $n_0 = n$, and $m_0 = m$. The halting problem for 2RMs is to decide, given a 2RM M , whether its computation on input $(0, 0)$ is finite (which implies that its last state is q_ℓ).

We convert a given 2RM M into a labeled GF^3 KB $(\mathcal{K}, \{a\}, \{b\})$, $\mathcal{K} = (\mathcal{O}, \mathcal{D})$ and signature Σ such that M halts iff $(\mathcal{K}, \{a\}, \{b\})$ is (non-)projectively $\text{GF}(\Sigma)$ -separable iff $(\mathcal{K}, \{a\}, \{b\})$ is (non-)projectively $\mathcal{ALC}(\Sigma)$ -separable. Let $M = (Q, P)$ with $Q = q_0, \dots, q_\ell$ and $P = I_0, \dots, I_{\ell-1}$. We assume w.l.o.g. that $\ell \geq 1$ and that if $I_i = -(p, q_j, q_k)$, then $q_j \neq q_k$. In \mathcal{K} , we use the following set of relation symbols:

- a binary symbol N connecting a configuration to its successor configuration;
- binary symbols R_1 and R_2 that represent the register contents via the length of paths;
- unary symbols q_0, \dots, q_ℓ representing the states of M ;
- a unary symbol S denoting points where a computation starts.
- a unary symbol D used to represent that there is some defect;
- binary symbols D_p^+, D_p^-, D_p^- used to describe defects in incrementing, decrementing, and keeping register $p \in \{0, 1\}$;
- ternary symbols $H_1^+, H_2^+, H_1^-, H_2^-, H_1^-, H_2^-$ used as guards for existential quantifiers.

The signature Σ consists of the symbols from the first four points above.

We define the ontology \mathcal{O} as the set of several GF^3 sentences.² The first sentence initializes the starting configuration:

$$\forall x(Sx \rightarrow (q_0x \wedge \neg \exists y R_0xy \wedge \neg \exists y R_1xy))$$

Second, whenever M is not in the final state, there is a next configuration with the correctly updated state. For $0 \leq i < \ell$, we include:

$$\begin{aligned} & \forall x(q_ix \rightarrow \exists y Nxy) \\ & \forall x(q_ix \rightarrow \forall y(Nxy \rightarrow q_jy)) \quad \text{if } I_i = +(p, q_j) \\ & \forall x((q_ix \wedge \neg \exists y R_pxy) \rightarrow \forall y(Nxy \rightarrow q_jy)) \quad \text{if } I_i = -(p, q_j, q_k) \\ & \forall x((q_ix \wedge \exists y R_pxy) \rightarrow \forall y(Nxy \rightarrow q_ky)) \quad \text{if } I_i = -(p, q_j, q_k) \end{aligned}$$

² The formulas that are not syntactically guarded can easily be rewritten into such formulas.

Moreover, if M is in the final state, there is no successor configuration:

$$\forall x(q_\ell x \rightarrow \neg \exists y Nxy).$$

The next conjunct expresses that either M does not halt or the representation of the computation of M contains a defect. It crucially uses non- Σ relation symbols. It takes the shape of

$$\forall x (Dx \rightarrow \exists y (Nxy \wedge \psi xy))$$

where ψxy is the following disjunction which ensures that there is a concrete defect (D_p^+ , D_p^- , D_p^\pm) here or some defect (D) in some successor state:

$$\begin{aligned} & D(y) \vee \\ & \bigvee_{I_i=+(p,q_j)} (q_i x \wedge q_j y \wedge (D_p^+ xy \vee D_{1-p}^- xy)) \vee \\ & \bigvee_{I_i=-(p,q_j,q_k)} (q_i x \wedge q_k y \wedge (D_p^- xy \vee D_{1-p}^- xy)) \vee \\ & \bigvee_{I_i=-(p,q_j,q_k)} (q_i x \wedge q_j y \wedge (D_p^\pm xy \vee D_{1-p}^\pm xy)) \end{aligned}$$

Finally, using the ternary symbols we make sure that the defects are realized, for example, by taking:

$$\begin{aligned} & \forall x \forall y (D_p^+ xy \rightarrow \\ & (\neg \exists z R_p yz \vee (\neg \exists z R_p xz \wedge \exists z (R_p yz \wedge \exists x R_p zx))) \vee \\ & \exists z (H_1^+ xyz \wedge R_p xz \wedge \exists x (H_2^+ xzy \wedge R_p yx \wedge D_p^+ zx))) \end{aligned}$$

Similar conjuncts implement the desired behaviour of D_p^\pm and D_p^- ; since they are constructed analogously to the last three lines above (but using guards H_j^- and H_j^\pm), details are omitted.

Finally, we define a database \mathcal{D} by taking

$$\mathcal{D} = \{S(a), D(a), S(b)\}.$$

Lemmas 4 and 5 below establish correctness of the reduction and thus Theorem 4.

Lemma 4. *If M halts, then there is an $\mathcal{ALC}(\Sigma)$ concept that non-projectively separates $(\mathcal{K}, \{a\}, \{b\})$.*

Proof. The idea is that the separating $\mathcal{ALC}(\Sigma)$ concept describes the halting computation of M , up to $\mathcal{ALC}(\Sigma)$ -bisimulations. More precisely, assume that M halts. We define an $\mathcal{ALC}(\Sigma)$ concept C such that $\mathcal{K} \models \neg C(a)$, but $\mathcal{K} \not\models \neg C(b)$. Intuitively, C represents the computation of M on input $(0,0)$, that is: if the computation is $(q_0, n_0, m_0), \dots, (q_k, n_k, m_k)$, then there is an N -path of length k (but not longer) such that any object reachable in $i \leq k$ steps from the beginning of the path is labeled with q_i , has an outgoing R_0 -path of length n_i and no longer

outgoing R_0 -path, and likewise for R_1 and m_i . In more detail, consider the Σ -structure \mathfrak{A} with

$$\begin{aligned} \text{dom}(\mathfrak{A}) = \{0, \dots, k\} \cup \{a_j^i \mid 0 < i \leq k, 0 < j < n_i\} \cup \\ \{b_j^i \mid 0 < i \leq k, 0 < j < m_i\} \end{aligned}$$

in which

$$\begin{aligned} N^{\mathfrak{A}} &= \{(i, i+1) \mid i < k\} \\ R_1^{\mathfrak{A}} &= \bigcup_{i \leq k} \{(i, a_1^i), (a_1^i, a_2^i), \dots, (a_{n_i-2}^i, a_{n_i-1}^i)\} \\ R_2^{\mathfrak{A}} &= \bigcup_{i \leq k} \{(i, b_1^i), (b_1^i, b_2^i), \dots, (b_{m_i-2}^i, b_{m_i-1}^i)\} \\ S^{\mathfrak{A}} &= \{0\} \\ q^{\mathfrak{A}} &= \{i \mid q_i = q\} \text{ for any } q \in Q. \end{aligned}$$

Then let C be the $\mathcal{ALC}(\Sigma)$ concept that describes \mathfrak{A} from the point of 0 up to $\mathcal{ALC}(\Sigma)$ -bisimulations. Clearly, $\mathcal{K} \cup \{C(b)\}$ is satisfiable. However, $\mathcal{K} \cup \{C(a)\}$ is unsatisfiable since the enforced computation does not contain a defect and cannot be extended to have one. In particular, there are no N -paths of length $> k$ in any model of $\mathcal{K} \cup \{C(a)\}$ and there are no defects in register updates in any model of $\mathcal{K} \cup \{C(a)\}$. \square

The following lemma implies that if M does not halt, then $(\mathcal{K}, \{a\}, \{b\})$ is neither projectively $\mathcal{L}(\Sigma)$ -separable nor non-projectively $\mathcal{L}(\Sigma)$ -separable for $\mathcal{L} = \text{GF}$ and in fact for every logic \mathcal{L} between GF and FO.

Lemma 5. *If M does not halt, then for every model \mathfrak{A} of \mathcal{K} , there is a model \mathfrak{B} of \mathcal{K} such that $(\mathfrak{A}, b^{\mathfrak{A}})$ is Γ -isomorphic to $(\mathfrak{B}, a^{\mathfrak{B}})$ where Γ consists of all symbols except $\text{sig}(\mathcal{O}) \setminus \Sigma$.*

Proof. Let \mathfrak{A} be a model of \mathcal{K} . We obtain \mathfrak{B} from \mathfrak{A} by re-interpreting $a^{\mathfrak{B}} = b^{\mathfrak{A}}$ and inductively defining the extensions of the symbols from

$$\text{sig}(\mathcal{O}) \setminus \Sigma = \{D, D_p^+, D_p^-, D_p^{\bar{-}}, H_1^+, H_2^+, H_1^-, H_2^-, H_1^{\bar{-}}, H_2^{\bar{-}}\}.$$

We start with $D^{\mathfrak{B}} = \{a^{\mathfrak{B}}\}$ and $X^{\mathfrak{B}} = \emptyset$ for all other symbols X from $\text{sig}(\mathcal{O}) \setminus \Sigma$. Then, whenever $d \in D^{\mathfrak{B}}$ we distinguish two cases:

- If there is an N -successor e of d such that the counters below d and e are not correctly updated with respect to the states at d, e , set the extensions of the symbols in $D_p^+, D_p^-, D_p^{\bar{-}}, H_1^+, H_2^+, H_1^-, H_2^-, H_1^{\bar{-}}, H_2^{\bar{-}}$ so as to represent the defect and finish the construction of \mathfrak{B} .
- Otherwise, choose an N -successor e of d and add e to $D^{\mathfrak{B}}$.

Note that, since M does not halt, we can always find such an N -successor as in the second item. \square

Let us now look at \mathcal{ALCFIO} .

Theorem 5. *Projective and non-projective \mathcal{L} -separability with signature are undecidable for every logic \mathcal{L} that contains \mathcal{ALCFIO} .*

The proof is by a reduction of the following undecidable tiling problem.

Definition 2. *A tiling system $S = (T, H, V, R, L, T, B)$ consists of a finite set T of tiles, horizontal and vertical matching relations $H, V \subseteq T \times T$, and sets $R, L, T, B \subseteq T$ of right tiles, left tiles, top tiles, and bottom tiles. A solution to S is a triple (n, m, τ) where $n, m \in \mathbb{N}$ and $\tau : \{0, \dots, n\} \times \{0, \dots, m\} \rightarrow T$ such that the following hold:*

1. $(\tau(i, j), \tau(i + 1, j)) \in H$, for all $i < n$ and $j \leq m$;
2. $(\tau(i, j), \tau(i, j + 1)) \in V$, for all $i \leq n$ and $j < m$;
3. $\tau(0, j) \in L$ and $\tau(n, j) \in R$, for all $j \leq m$;
4. $\tau(i, 0) \in B$ and $\tau(i, m) \in T$, for all $i \leq n$.

We show how to convert a tiling system S into a labeled \mathcal{ALCFIO} -KB (\mathcal{K}, P, N) and signature Σ such that S has a solution iff (\mathcal{K}, P, N) is $\mathcal{ALCFIO}(\Sigma)$ -separable iff (\mathcal{K}, P, N) is projectively $\mathcal{ALCFIO}(\Sigma)$ -separable.

Let $S = (T, H, V, R, L, T, B)$ be a tiling system. Define an ontology \mathcal{O} that consists of the following statements:

- The roles r_x, r_y , and their inverses are functional:

$$\top \sqsubseteq (\leq 1 r), \text{ for } r \in \{r_x, r_y, r_x^-, r_y^-\}$$

- Every grid node is labeled with exactly one tile and the matching conditions are satisfied:

$$\begin{aligned} \top &\sqsubseteq \bigsqcup_{t \in T} (t \sqcap \bigsqcap_{t' \in T, t' \neq t} \neg t') \\ \top &\sqsubseteq \bigsqcap_{t \in T} (t \rightarrow (\bigsqcup_{(t, t') \in H} \forall r_x. t' \sqcap \bigsqcup_{(t, t') \in V} \forall r_y. t')) \end{aligned}$$

- The concepts **left**, **right**, **top**, **bottom** mark the borders of the grid in the expected way:

$$\begin{aligned} \text{right} &\sqsubseteq \neg \exists r_x. \top \sqcap \forall r_y. \text{right} \sqcap \forall r_y^{-1}. \text{right} \\ \neg \text{right} &\sqsubseteq \exists r_x. \top \end{aligned}$$

and similarly for **left**, **top**, and **bottom**.

- The individual name o marks the origin:

$$\{o\} \sqsubseteq \text{left} \sqcap \text{bottom}.$$

- there is no infinite outgoing r_x/r_y -path starting at o and grid cells close in the part of models reachable from o :

$$\begin{aligned} Q &\sqsubseteq \exists r_x. Q \sqcup \exists r_y. Q \sqcup (\exists r_x. \exists r_y. P \sqcap \exists r_y. \exists r_x. \neg P) \\ A_1 \sqcap A_2 &\sqsubseteq \exists u. (\{o\} \sqcap Q) \end{aligned}$$

The final item deserves some further explanation. It is to be read as follows: the stated properties hold in a model \mathfrak{A} whenever \mathfrak{A} can *not* be extended to a model of the upper CI that makes true Q at o . In conjunction with the database, the second CI is a switch that will allow us to sometimes require that Q is made true at o .

Set $\Sigma = T \cup \{r_x, r_y, \text{left}, \text{right}, \text{top}, \text{bottom}\}$ and consider the labeled KB $(\mathcal{K}, \{a\}, \{b\})$ where $\mathcal{K} = (\mathcal{O}, \mathcal{D})$ with $\mathcal{D} = \{A_1(a), Y(b)\}$ with Y a fresh (dummy) concept name.

Lemma 6. *If S has a solution, then there is an $\mathcal{ALC}\mathcal{IO}(\Sigma)$ concept that non-projectively separates $(\mathcal{K}, \{a\}, \{b\})$.*

Proof. We design the $\mathcal{ALC}\mathcal{I}(\Sigma)$ concept $C = \neg D$ so that any model of D and \mathcal{O} , even without the CIs from the last item, includes a properly tiled $n \times m$ -grid with lower left corner o .

For every word $w \in \{r_x, r_y\}^*$, denote by \overleftarrow{w} the word that is obtained by reversing w and then adding $\bar{\cdot}$ to each symbol. Let $|w|_r$ denote the number of occurrences of the symbol r in w . Now, $D = A_2 \sqcap \exists u. E$ where E is the conjunction of

$$\{o\} \sqcap \forall r_x^n. \text{right} \sqcap \forall r_y^m. \text{top}$$

and for every $w \in \{r_x, r_y\}^*$ such that $|w|_{r_x} < n$ and $|w|_{r_y} < m$, the concept

$$\exists(w \cdot r_x r_y r_x^- r_y^- \cdot \overleftarrow{w}). \{o\},$$

where $\exists w.F$ abbreviates $\exists r_1 \dots \exists r_k.F$ if $w = r_1 \dots r_k$. It is readily checked that E (and thus D) indeed enforces a properly tiled grid as announced. Then, due to the CIs in the last item, $\mathcal{K} \cup \{D(a)\}$ is unsatisfiable: Any model \mathfrak{A} has to satisfy $a^{\mathfrak{A}} \in A_1^{\mathfrak{A}}$ since $A_1(a) \in \mathcal{D}$ and $a^{\mathfrak{A}} \in A_2^{\mathfrak{A}}$, due to the assertion $D(a)$. Hence the last CIs become ‘active’, which is in conflict to the fact that the model enforced by D contains neither an infinite r_x/r_y -path nor a non-closing grid-cell. Thus, $\mathcal{K} \models C(a)$ as required.

Now for $\mathcal{K} \not\models C(b)$. We find a model \mathfrak{A} of \mathcal{K} with $b^{\mathfrak{A}} \in D^{\mathfrak{A}}$ since all CIs in \mathcal{O} except this from the last item are satisfied by the grid enforced by D and the CIs in the last item can be made ‘inactive’ by making A_1 false at $b^{\mathfrak{A}}$. \square

The following lemma implies that if S has no solution, then $(\mathcal{K}, \{a\}, \{b\})$ is neither projectively $\mathcal{L}(\Sigma)$ -separable nor non-projectively $\mathcal{L}(\Sigma)$ -separable for $\mathcal{L} = \mathcal{ALC}\mathcal{IO}$ and in fact for every logic \mathcal{L} between $\mathcal{ALC}\mathcal{IO}$ and FO.

Lemma 7. *If S has no solution, then for every model \mathfrak{A} of \mathcal{K} , there is a model \mathfrak{B} of \mathcal{K} such that $(\mathfrak{A}, b^{\mathfrak{A}})$ is Γ -isomorphic to $(\mathfrak{B}, a^{\mathfrak{A}})$ where Γ consists of all symbols except $\{u, A_1, Q, P\}$.*

Proof. (sketch) If $b^{\mathfrak{A}} \notin A_2^{\mathfrak{A}}$, then we can simply obtain \mathfrak{B} from \mathfrak{A} by switching $a^{\mathfrak{A}}$ and $b^{\mathfrak{A}}$ and making A_1 true at $a^{\mathfrak{A}}$. If $b^{\mathfrak{A}} \in A_2^{\mathfrak{A}}$, then after switching we additionally have to re-interpret Q , P , and u in a suitable way. But S has no solution and thus when following r_x/r_y -paths from o in \mathcal{I} , we must either encounter an infinite such path or a non-closing grid cell as otherwise we can extract from \mathcal{I} a solution for S . Thus we can re-interpret Q , P , and u as required. \square

6 Strong Separability with Signature

We introduce strong separability of labeled KBs. The crucial difference to weak separability is that the negation of the separating formula must be entailed at all negative examples.

Definition 3. Let (\mathcal{K}, P, N) be a labeled FO-KB and $\Sigma \subseteq \text{sig}(\mathcal{K})$ a signature. An FO-formula $\varphi(\mathbf{x})$ strongly Σ -separates (\mathcal{K}, P, N) if $\text{sig}(\mathcal{K}) \cap \text{sig}(\varphi) \subseteq \Sigma$ and

1. $\mathcal{K} \models \varphi(\mathbf{a})$ for all $\mathbf{a} \in P$ and
2. $\mathcal{K} \models \neg\varphi(\mathbf{a})$ for all $\mathbf{a} \in N$.

Let \mathcal{L}_S be a fragment of FO. We say that (\mathcal{K}, P, N) is strongly projectively $\mathcal{L}_S(\Sigma)$ -separable if there exists an $\mathcal{L}_S(\Sigma)$ -formula $\varphi(\mathbf{x})$ that strongly separates (\mathcal{K}, P, N) and non-projectively strongly $\mathcal{L}_S(\Sigma)$ -separable if there is such a $\varphi(\mathbf{x})$ with $\text{sig}(\varphi) \subseteq \Sigma$.

In contrast to weak separability, any formula φ that strongly separates a labeled KB (\mathcal{K}, P, N) and uses helper symbols R that are not in Σ can easily be transformed into a strongly separating formula that uses only symbols from Σ : simply replace any such R by a relation symbol R' of the same arity that is in Σ . Then, if φ strongly separates (\mathcal{K}, P, N) , so does the resulting formula φ' . If no relation symbol of the same arity as R occurs in Σ one can alternatively replace relevant subformulas by \top or \perp . In what follows, we thus only consider non-projective strong separability and simply speak of strong separability.

Note that for languages \mathcal{L}_S closed under conjunction and disjunction a labeled KB (\mathcal{K}, P, N) is strongly $\mathcal{L}_S(\Sigma)$ -separable iff every $(\mathcal{K}, \{\mathbf{a}\}, \{\mathbf{b}\})$ with $\mathbf{a} \in P$ and $\mathbf{b} \in N$ is strongly $\mathcal{L}_S(\Sigma)$ -separable. In fact, if $\varphi_{\mathbf{a},\mathbf{b}}$ strongly separates $(\mathcal{K}, \{\mathbf{a}\}, \{\mathbf{b}\})$ for $\mathbf{a} \in P$ and $\mathbf{b} \in N$, then $\bigvee_{\mathbf{a} \in P} \bigwedge_{\mathbf{b} \in N} \varphi_{\mathbf{a},\mathbf{b}}$ strongly separates (\mathcal{K}, P, N) . Without loss of generality, we may thus work with labeled KBs with singleton sets of positive and negative examples.

Each choice of an ontology language \mathcal{L} and a separation language \mathcal{L}_S thus gives rise to a (single) strong separability problem that we refer to as *strong $(\mathcal{L}, \mathcal{L}_S)$ -separability*, defined in the expected way:

PROBLEM : strong $(\mathcal{L}, \mathcal{L}_S)$ separability with signature
INPUT : labeled \mathcal{L} -KB (\mathcal{K}, P, N) and signature $\Sigma \subseteq \text{sig}(\mathcal{K})$
QUESTION : Is (\mathcal{K}, P, N) strongly $\mathcal{L}_S(\Sigma)$ -separable?

If $\mathcal{L} = \mathcal{L}_S$, then we simply speak of strong \mathcal{L} -separability. The study of strong separability is very closely linked to the study of interpolants and the Craig interpolation property. Given formulas $\varphi(\mathbf{x}), \psi(\mathbf{x})$ and a fragment \mathcal{L} of FO, we say that an \mathcal{L} -formula $\chi(\mathbf{x})$ is an \mathcal{L} -interpolant of φ, ψ if $\varphi(\mathbf{x}) \models \chi(\mathbf{x})$, $\chi(\mathbf{x}) \models \psi(\mathbf{x})$, and $\text{sig}(\chi) \subseteq \text{sig}(\varphi) \cap \text{sig}(\psi)$. We say that \mathcal{L} has the CIP if for any \mathcal{L} -formulas $\varphi(\mathbf{x}), \psi(\mathbf{x})$ such that $\varphi(\mathbf{x}) \models \psi(\mathbf{x})$ there exists an \mathcal{L} -interpolant of φ, ψ . FO has the CIP, and so does GNF [10, 8], at least if one admits non-shared constants in the interpolant. On the other hand, GF does not have the CIP [24].

The link between the interpolants, the CIP, and strong separability is easy to see: assume a labeled FO-KB $(\mathcal{K}, \{\mathbf{a}\}, \{\mathbf{b}\})$ with $\mathcal{K} = (\mathcal{O}, \mathcal{D})$ and a signature $\Sigma \subseteq \text{sig}(\mathcal{K})$ are given. Obtain $\mathcal{K}_{\Sigma, \mathbf{a}}$ and $\mathcal{K}_{\Sigma, \mathbf{b}}$ from \mathcal{K} by

- replacing all non- Σ -relation symbols R in \mathcal{K} by fresh symbols $R^{\mathbf{a}}$ and $R^{\mathbf{b}}$, respectively;
- replacing all constant symbols c by fresh variables $x_{c, \mathbf{a}}$ and $x_{c, \mathbf{b}}$ which are distinct, except that \mathbf{a} and \mathbf{b} are replaced by the same tuple \mathbf{x} in $\mathcal{K}_{\Sigma, \mathbf{a}}$ and $\mathcal{K}_{\Sigma, \mathbf{b}}$, respectively.

Then let $\varphi_{\Sigma, \mathbf{a}}(\mathbf{x}) = \exists \mathbf{z}(\bigwedge \mathcal{K}_{\Sigma, \mathbf{a}})$, where \mathbf{z} is the sequence of free variables in $\mathcal{K}_{\Sigma, \mathbf{a}}$ without the variables in \mathbf{x} and $(\bigwedge \mathcal{K}_{\Sigma, \mathbf{a}})$ is the conjunction of all formulas in $\mathcal{K}_{\Sigma, \mathbf{a}}$. $\varphi_{\Sigma, \mathbf{b}}(\mathbf{x})$ is defined in the same way, with \mathbf{a} replaced by \mathbf{b} . The following lemma is a direct consequence of the construction.

Lemma 8. *Let \mathcal{L} be a fragment of FO. Then the following conditions are equivalent for any formula φ in \mathcal{L} :*

1. φ strongly $\mathcal{L}(\Sigma)$ -separates $(\mathcal{K}, \{\mathbf{a}\}, \{\mathbf{b}\})$;
2. φ is an \mathcal{L} -interpolant for $\varphi_{\Sigma, \mathbf{a}}(\mathbf{x}), \neg \varphi_{\Sigma, \mathbf{b}}(\mathbf{x})$.

Thus, the problem whether a labeled KB (\mathcal{K}, P, N) is strongly $\mathcal{L}(S)$ -separable and the computation of a strongly separating formula can be equivalently formulated as an interpolant existence problem. As FO has the CIP, we obtain the following characterization of the existence of strongly FO(Σ)-separating formulas.

Theorem 6. *The following conditions are equivalent for any labeled FO-KB $(\mathcal{K}, \{\mathbf{a}\}, \{\mathbf{b}\})$ and signature $\Sigma \subseteq \text{sig}(\mathcal{K})$:*

1. $(\mathcal{K}, \{\mathbf{a}\}, \{\mathbf{b}\})$ is strongly FO(Σ)-separable;
2. $\varphi_{\Sigma, \mathbf{a}}(\mathbf{x}) \models \neg \varphi_{\Sigma, \mathbf{b}}(\mathbf{x})$.

For fragments \mathcal{L} of FO such as \mathcal{ALCI} , GF, and GNF, Lemma 8 has to be applied with some care, as one has to ensure that the formulas $\varphi_{\Sigma, \mathbf{a}}(\mathbf{x}), \neg \varphi_{\Sigma, \mathbf{b}}(\mathbf{x})$ are still within \mathcal{L} . This will be discussed in the next two sections.

7 Strong Separability in \mathcal{ALCI}

We first compare the strong separating power of \mathcal{ALCI} with signature restrictions to the strong separating power of FO with signature restrictions and show that they differ. This is in contrast to strong separability without signature restrictions. We then show that strong \mathcal{ALCI} -separability with signature restrictions is 2EXPTIME-complete, thus one exponential harder than strong \mathcal{ALCI} -separability without signature restrictions. Observe that we cannot apply the CIP of \mathcal{ALCI} [15] to investigate strong separability for \mathcal{ALCI} -KBs as one cannot encode the atomic formulas of the database in \mathcal{ALCI} .³

³ One could instead move to the extension \mathcal{ALCIO} of \mathcal{ALCI} with nominals. This language, however, does not have the CIP [15]. Recently, interpolant existence in \mathcal{ALCIO} has been investigated in [4], and the results could be applied here. The following direct approach is of independent value, however.

In [26], strong separability is studied without signature restrictions. It turned that a labeled \mathcal{ALCI} -KB is strongly \mathcal{ALCI} -separable without signature restrictions iff it is strongly FO-separable without signature restrictions. Unfortunately, this is not the case with signature restrictions. A simple counterexample is given in the following example.

Example 5. Let $\mathcal{D} = \{R(a, a), A(b)\}$ and $\mathcal{O} = \{A \sqsubseteq \forall R. \neg A\}$. Let $\mathcal{K} = (\mathcal{O}, \mathcal{D})$ and $\Sigma = \{R\}$. Then $R(x, x)$ strongly separates $(\mathcal{K}, \{a\}, \{b\})$ and thus $(\mathcal{K}, \{a\}, \{b\})$ is strongly FO(Σ)-separable. The characterization below immediately implies that $(\mathcal{K}, \{a\}, \{b\})$ is not strongly $\mathcal{ALCI}(\Sigma)$ -separable.

We now show that strong \mathcal{ALCI} -separability with signature restrictions is 2EXPTIME-complete. To this end we first give a model-theoretic characterization of strong \mathcal{ALCI} -separability using \mathcal{ALCI} -bisimulations.

Theorem 7. *Let $(\mathcal{K}, \{a\}, \{b\})$ be an \mathcal{ALCI} -KB and $\Sigma \subseteq \text{sig}(\mathcal{K})$ a signature. Then the following conditions are equivalent:*

1. $(\mathcal{K}, \{a\}, \{b\})$ is strongly $\mathcal{ALCI}(\Sigma)$ -separable;
2. There are no models \mathfrak{A} and \mathfrak{B} of \mathcal{K} such that $\mathfrak{A}, a^{\mathfrak{A}} \sim_{\mathcal{ALCI}, \Sigma} \mathfrak{B}, b^{\mathfrak{B}}$.

The proof is straightforward using Lemma 2. By working with isomorphic copies of the database \mathcal{D} it thus suffices to show the following result.

Lemma 9. *Let $(\mathcal{K}, \{a\}, \{b\})$ be a labeled \mathcal{ALCI} -KB with $\mathcal{K} = (\mathcal{O}, \mathcal{D})$ such that a, b are in distinct maximal connected components of \mathcal{D} . Then the problem to decide whether there exists a model \mathfrak{A} of \mathcal{K} such that $\mathfrak{A}, a^{\mathfrak{A}} \sim_{\mathcal{ALCI}, \Sigma} \mathfrak{A}, b^{\mathfrak{A}}$ is 2EXPTIME-complete.*

Let $\mathcal{K} = (\mathcal{O}, \mathcal{D})$ and let $\Sigma \subseteq \text{sig}(\mathcal{K})$ be a signature. We start with proving the upper bound and use the notion of \mathcal{K} -types as introduced in Section 4. Let R be a role. We say that \mathcal{K} -types t_1 and t_2 are R -coherent if there exists a model \mathfrak{A} of \mathcal{O} and nodes d_1 and d_2 realizing t_1 and t_2 , respectively, such that $(d_1, d_2) \in R^{\mathfrak{A}}$. We write $t_1 \rightsquigarrow_R t_2$ in this case.

Definition 4 ((\mathcal{O}, Σ)-amalgamable). *A set Φ of \mathcal{K} -types is (\mathcal{O}, Σ)-amalgamable if there exist models \mathfrak{A}_t of \mathcal{O} for $t \in \Phi$ with elements d_t realizing t in \mathfrak{A}_t such that all \mathfrak{A}_t, d_t with $t \in \Phi$ are $\mathcal{ALCI}(\Sigma)$ -bisimilar.*

Lemma 10. *The set of all (\mathcal{O}, Σ)-amalgamable sets of \mathcal{K} -types can be computed in double exponential time.*

We devise an elimination procedure as follows. Start with M_0 the set of all sets of \mathcal{K} -types. Given a set M_i of sets of types, we obtain M_{i+1} by eliminating all $\Phi = \{t_1, \dots, t_n\}$ from M_i which do not satisfy the following conditions:

1. for every $A \in \Sigma$, we have $A \in t_i$ iff $A \in t_j$, for all $t_i, t_j \in \Phi$;
2. for every t_i , every Σ -role R , and every $\exists R.C \in t_i$ there are \mathcal{K} -types t'_1, \dots, t'_n such that $C \in t'_i$ and $t_j \rightsquigarrow_R t'_j$, for all j , and $\{t'_1, \dots, t'_n\} \in M_i$.

Let M^* be where the sequence M_0, M_1, \dots stabilizes.

Claim. $\Phi \in M^*$ iff Φ is (\mathcal{O}, Σ) -amalgamable.

Proof of the Claim. For the “if”-direction, suppose that $\Phi = \{t_1, \dots, t_n\}$ is (\mathcal{O}, Σ) -amalgamable. We can fix (disjoint) models $\mathfrak{A}_{t_1}, \dots, \mathfrak{A}_{t_n}$ of \mathcal{O} realizing types t_i at d_{t_i} . Let \mathfrak{A} denote the union of $\mathfrak{A}_{t_1}, \dots, \mathfrak{A}_{t_n}$ and let S be the set of all pairs (d, e) which are Σ -bisimilar in \mathfrak{A} . Recall that S is an equivalence relation. By assumption, we have $(d_{t_i}, d_{t_j}) \in S$, for all i, j . It can be verified that the set N defined by

$$N = \{\{\text{tp}_{\mathcal{K}}(\mathfrak{A}, d) \mid (d, e) \in S\} \mid e \in \text{dom}(\mathfrak{A})\}$$

is contained in all M_i and thus in M^* .

For “only if”, let $\Phi = \{t_1, \dots, t_n\} \in M^*$. We inductively construct a domain Δ , a map π of the domain Δ to \mathcal{K} -types, and an equivalence relation S . During the construction, we preserve the invariant

(*) $\pi(D) = \{\pi(d) \mid d \in D\} \in M^*$, for every $D \in S$.

For the construction, start with setting

$$- \Delta_0 = \{d_1, \dots, d_n\}, \pi(d_i) = t_i, \text{ for all } i, \text{ and } S = \{\Delta_0\}.$$

Obviously, the invariant is satisfied. To obtain Δ_{i+1} from Δ_i , choose some $D = \{e_1, \dots, e_m\} \in S$, some $\exists R.C \in \pi(d_i)$ for some Σ -role R . By the invariant, we have $\{t_1, \dots, t_k\} = \pi(D) \in M^*$. Let t'_1, \dots, t'_k be the types that exist due to **(E2)**. Now, add fresh elements $e_1 R e'_1, \dots, e_m R e'_m$ to Δ_i , set $\pi(e_i R e'_i) = \pi(e_i)'$, for all i , and add $\{e_1 R e'_1, \dots, e_m R e'_m\}$ to S . By construction, the invariant (*) is preserved.

Now define a structure \mathfrak{A} by taking:

$$\begin{aligned} \text{dom}(\mathfrak{A}) &= \bigcup_{i \geq 0} \Delta_i \\ A^{\mathfrak{A}} &= \{e \mid A \in \pi(e)\} \\ r^{\mathfrak{A}} &= \{(d, d R e) \mid d R e \in \text{dom}(\mathfrak{A})\} \cup \\ &\quad \{(d R^- e, d) \mid d R^- e \in \text{dom}(\mathfrak{A})\} \end{aligned}$$

By construction, S is an $\mathcal{ALCT}(\Sigma)$ -bisimulation that contains (d_i, d_j) for all i, j . Since \mathcal{K} -types are realizable by definition, we can extend \mathfrak{A} to a model \mathfrak{A}^* of \mathcal{O} by adding non- Σ -subtrees whenever they are needed.

It follows that Φ is (\mathcal{O}, Σ) -amalgamable. This finishes the proof of the Claim.

It remains to discuss the running time of the algorithm. The initial set M_0 contains at most double exponentially many elements. Since in every round some element is removed from M_i , the stabilization is reached after $|M_0|$ rounds. It remains to observe that the elimination conditions 1 and 2 can be checked in double exponential time.

It is important to note that the proof of Lemma 10 shows that, if a set Φ is (\mathcal{O}, Σ) -amalgamable, then this is witnessed by disjoint tree-shaped models \mathfrak{A}_t

with root d_t , for each $t \in \Phi$, and $\mathcal{ALCI}(\Sigma)$ -bisimulations S which “never visit the roots again”, that is, if $(d_t, e) \in S$ or $(e, d_t) \in S$, then $e = d_{t'}$ for some t' . This will be used in the proof of the characterization below.

Let Ψ be a mapping associating with every $c \in \text{cons}(\mathcal{D})$ a \mathcal{K} -type t_c and a set Φ_c of \mathcal{K} -types. We say that Ψ is \mathcal{K}, a, b -satisfiable if

1. there exists a model \mathfrak{A} of \mathcal{K} realizing t_c in $c^{\mathfrak{A}}$ for $c \in \text{cons}(\mathcal{D})$;
2. $\Phi_c \cup \{t_c\}$ is (\mathcal{O}, Σ) -amalgamable, for all $c \in \text{cons}(\mathcal{D})$;
3. $\Phi_a \cup \Phi_b \cup \{t_a, t_b\}$ is (\mathcal{O}, Σ) -amalgamable;
4. If $R(d, e) \in \mathcal{O}$, for some Σ -role R , and $t \in \Phi_d$, then there exists $t' \in \Phi_e$ such that $t \rightsquigarrow_R t'$;

Lemma 11. *The following conditions are equivalent:*

- There exists a model \mathfrak{A} of \mathcal{K} such that $\mathfrak{A}, a^{\mathfrak{A}} \sim_{\mathcal{ALCI}, \Sigma} \mathfrak{A}, b^{\mathfrak{A}}$.
- There exists Ψ that is \mathcal{K}, a, b -satisfiable.

For “only if”, let \mathfrak{A} be a model of \mathcal{K} such that $(a^{\mathfrak{A}}, b^{\mathfrak{A}}) \in S$. Let t_c be the type realized in $c^{\mathfrak{A}}$ for $c \in \text{cons}(\mathcal{D})$ and let Φ_c be the set of all \mathcal{K} -types realized in nodes that are $\mathcal{ALCI}(\Sigma)$ -bisimilar in \mathfrak{A} to $c^{\mathfrak{A}}$, that is,

$$\Phi_c = \{\text{tp}_{\mathcal{K}}(\mathfrak{A}, d) \mid \mathfrak{A}, c^{\mathfrak{A}} \sim_{\mathcal{ALCI}, \Sigma} \mathfrak{A}, d^{\mathfrak{A}}\}.$$

It is easy to see that the resulting Ψ is as required.

Conversely, assume Ψ is given. Due to Condition 1, we can fix a model \mathfrak{B} of \mathcal{K} realizing t_c in $c^{\mathfrak{B}}$ for $c \in \text{cons}(\mathcal{D})$. Moreover, due to Condition 2, we can fix for any $c \in \text{cons}(\mathcal{D})$ and $t \in \Phi_c \cup \{t_c\}$ tree-shaped models $\mathfrak{A}_{c,t}$ of \mathcal{O} with root $d_{c,t}$ such that all pointed structures in

$$\{\mathfrak{A}_{c,t}, d_{c,t} \mid t \in \Phi_c \cup \{t_c\}\}$$

are $\mathcal{ALCI}(\Sigma)$ -bisimilar. By Condition 3, we can assume that $\mathfrak{A}_{a,t_a}, d_{a,t_a}$ and $\mathfrak{A}_{b,t_b}, d_{b,t_b}$ are $\mathcal{ALCI}(\Sigma)$ -bisimilar.

We inductively construct a model \mathfrak{A} and a $\mathcal{ALCI}(\Sigma)$ -bisimulation S . We start with the structure \mathfrak{A}_0 which is obtained as follows. Let \mathfrak{B}' be \mathfrak{B} restricted to the domain $\{d^{\mathfrak{B}} \mid d \in \text{cons}(\mathcal{D})\}$. Now, \mathfrak{A}_0 is the union of \mathfrak{B}' and all structures \mathfrak{A}_{c,t_c} for all $c \in \text{cons}(\mathcal{D})$, always identifying the root of \mathfrak{A}_{c,t_c} with $c^{\mathfrak{B}}$. Moreover, let S_0 be the $\mathcal{ALCI}(\Sigma)$ -bisimulation between \mathfrak{A}_{a,t_a} and \mathfrak{A}_{b,t_b} . By the comment after the proof of Lemma 10, $(a^{\mathfrak{A}_0}, b^{\mathfrak{A}_0})$ is the only tuple in S_0 that contains $a^{\mathfrak{A}_0}$ or $b^{\mathfrak{A}_0}$.

Note that \mathfrak{A}_0 is in fact a model of \mathcal{K} but S_0 is not yet a bisimulation. In order to make it one, we “chase” the database in both connected components preserving the following invariant (which is obviously satisfied for \mathfrak{A}_0, S_0):

- (*) If $(c, d^{\mathfrak{A}_i}) \in S_i$ or $(d^{\mathfrak{A}_i}, c) \in S_i$ for some $d \in \text{cons}(\mathcal{D})$, then the type t realized by c in \mathfrak{A}_i satisfies $t \in \Phi_d$. Moreover, the trees below c and $d^{\mathfrak{A}_i}$ are $\mathcal{ALCI}(\Sigma)$ -bisimilar.

In the inductive step, obtain $\mathfrak{A}_{i+1}, S_{i+1}$ from \mathfrak{A}_i, S_i by applying one of the following rules:

- Choose $(c, d^{2^i}) \in S_i$ and e such that $R(d, e) \in \mathcal{D}$, and let $t = \text{tp}_{\mathfrak{A}_i}(c)$ be the type of c realized in \mathfrak{A}_i . By (*), we know that $t \in \Phi_d$. By Condition 4, we can choose $t' \in \Phi_e$ with $t \rightsquigarrow_R t'$. Now, add a copy of $\mathfrak{A}_{e,t'}$ to \mathfrak{A}_i and make its root an R -successor of c . By Condition 2, there is an $\mathcal{ALCI}(\Sigma)$ -bisimulation S between $\mathfrak{A}_{e,t'}$ and the tree \mathfrak{A}_{e,t_e} below $e_i^{2^i}$. Set $S_{i+1} = S_i \cup S$.
- Choose $(d^{2^i}, c) \in S_i$ and e such that $R(d, e) \in \mathcal{D}$, and proceed analogously to the first rule.

Let $\mathfrak{A} = \bigcup \mathfrak{A}_i$ and $S = \bigcup S_i$.

Claim. \mathfrak{A} is a model of \mathcal{K} and S is $\mathcal{ALCI}(\Sigma)$ -bisimulation with $(a^{\mathfrak{A}}, b^{\mathfrak{A}}) \in S$.

Proof of the Claim. We have $\mathfrak{A} \models \mathcal{K}$ since $\mathfrak{A}_i \models \mathcal{K}$, for all i . Moreover, $(a^{\mathfrak{A}}, b^{\mathfrak{A}}) \in S$ since $(a^{\mathfrak{A}}, b^{\mathfrak{A}}) \in S_0$, by definition of S_0 . To see that S is an $\mathcal{ALCI}(\Sigma)$ -bisimulation, let $(d, e) \in S$. We distinguish two cases:

- None of d, e is in $\{c^{2^i} \mid c \in \text{cons}(\mathcal{D})\}$. Thus, $(d, e) \in S$ because d and e are inner nodes of some of the trees $\mathfrak{A}_{c,t}$ that were fixed in the beginning. By construction, (d, e) is an element of a $\mathcal{ALCI}(\Sigma)$ -bisimulation S' between those trees. Thus, for every R -successor d' of d in \mathfrak{A} , R a Σ -role, there is an R -successor e' of e with $(d', e') \in S'$ and thus $(d', e') \in S$. The forth-condition is analogous.
- One of d, e is in $\{c^{2^i} \mid c \in \text{cons}(\mathcal{D})\}$, say $d = f^{2^i}$. Suppose d' is an R -successor of d , for some Σ -role R . We distinguish two cases:
 - d' is in the subtree \mathfrak{A}_{d,t_d} below d . Then because of (*), there is an R -successor e' of e in the tree below e such that $(d', e') \in S$.
 - $d' = g^{2^i}$ for some $R(f, g) \in \mathcal{D}$. Since the rules are applied exhaustively, there is an R -successor e' of e in \mathfrak{A} such that $(d', e) \in S$.

The forth-condition is analogous.

This finishes the proof of the Claim and, in fact, of the Lemma.

We can thus use the following algorithm to decide strong $\mathcal{ALCI}(\Sigma)$ -separability on input (\mathcal{K}, P, N) .

1. compute the set of all (\mathcal{O}, Σ) -amalgamable sets.
2. for all $a \in P$ and $b \in P$:
 - (a) enumerate all possible mappings Ψ consisting of \mathcal{K} -types t_c and sets of \mathcal{K} -types Φ_c , for every $c \in \text{cons}(\mathcal{D})$.
 - (b) if Ψ is \mathcal{K}, a, b -satisfiable, that is, satisfies Conditions 1–4 above, return “not separable.”
3. return “separable”.

The algorithm is correct due to Theorem 7 and Lemma 11. Moreover, it runs in double exponential time since Step 1 can be executed in double exponential time, by Lemma 10, there are only double exponentially many possible mappings Ψ , and \mathcal{K}, a, b -satisfiability can be checked in double exponential time: Condition 1

can be done in exponential time, Conditions 2 and 3 are a mere lookup in the (precomputed) amalgamable sets, and Condition 4 can be tested in double exponential time.

For the 2EXPTIME lower bound, we reduce the word problem for exponentially space bounded alternating Turing machines (ATMs). We actually use a slightly unusual ATM model which is easily seen to be equivalent to the standard model.

An *alternating Turing machine (ATM)* is a tuple $M = (Q, \Theta, \Gamma, q_0, \Delta)$ where $Q = Q_{\exists} \uplus Q_{\forall}$ is the set of states that consists of *existential states* in Q_{\exists} and *universal states* in Q_{\forall} . Further, Θ is the input alphabet and Γ is the tape alphabet that contains a *blank symbol* $\square \notin \Theta$, $q_0 \in Q_{\exists}$ is the *starting state*, and the *transition relation* Δ is of the form $\Delta \subseteq Q \times \Gamma \times Q \times \Gamma \times \{L, R\}$. The set $\Delta(q, a) := \{(q', a', M) \mid (q, a, q', a', M) \in \Delta\}$ must contain exactly two or zero elements for every $q \in Q$ and $a \in \Gamma$. Moreover, the state q' must be from Q_{\forall} if $q \in Q_{\exists}$ and from Q_{\exists} otherwise, that is, existential and universal states alternate. Note that there is no accepting state. The ATM accepts if it runs forever and rejects otherwise. Starting from the standard ATM model, this can be achieved by assuming that exponentially space bounded ATMs terminate on any input and then modifying them to enter an infinite loop from the accepting state.

A *configuration* of an ATM is a word wqw' with $w, w' \in \Gamma^*$ and $q \in Q$. We say that wqw' is *existential* if q is, and likewise for *universal*. *Successor configurations* are defined in the usual way. Note that every configuration has exactly two successor configurations.

A *computation tree* of an ATM M on input w is an infinite tree whose nodes are labeled with configurations of M such that

- the root is labeled with the initial configuration q_0w ;
- if a node is labeled with an existential configuration wqw' , then it has a single successor and this successor is labeled with a successor configuration of wqw' ;
- if a node is labeled with a universal configuration wqw' , then it has two successors and these successors are labeled with the two successor configurations of wqw' .

An ATM M *accepts* an input w if there is a computation tree of M on w .

We reduce the word problem for 2^n -space bounded ATMs which is known to be 2EXPTIME-hard [16]. The idea of the reduction is as follows. We set

$$\begin{aligned} \mathcal{D} &= \{A(a), r(b, b), B(b)\}, \\ \Sigma &= \{r, s, Z, B_{\forall}, B_{\exists}^1, B_{\exists}^2\} \cup \{A_{\sigma} \mid \sigma \in \Gamma \cup (Q \times \Gamma)\} \end{aligned}$$

The ontology \mathcal{O} enforces that in an A -node starts an infinite r -path ρ . Along ρ , a counter counts modulo 2^n using concept names not in Σ . In each point of ρ starts an infinite tree along role s that is supposed to mimick the computation tree of M . Along this tree, two counters are maintained:

- one counter starting at 0 and counting modulo 2^n to divide the tree in subpaths of length 2^n ; each such path of length 2^n represents a configuration;

- another counter starting at the value of the counter along ρ and also counting modulo 2^n .

To link successive configurations we use that if $(\mathcal{K}, \{a\}, \{b\})$ has no strong $\mathcal{ALCI}(\Sigma)$ solution, then there exist models \mathfrak{A} and \mathfrak{B} of \mathcal{K} such that $\mathfrak{A}, a^{\mathfrak{A}} \sim_{\Sigma} \mathfrak{B}, b^{\mathfrak{B}}$: from $r(b, b) \in \mathcal{D}$ it follows that in \mathfrak{A} all nodes on the r -path ρ are Σ -bisimilar. Thus, each node on the ρ is the starting point of s -trees with identical Σ -decorations. As on the m th s -tree the second counter starts at all nodes at distances $k \times 2^n - m$, for all $k \geq 1$, we are in the position to coordinate all positions at all successive configurations.

The ontology \mathcal{O} is constructed as follows. We first enforce the infinite r -path ρ with the counter, which is realized using concept names $A_i, \bar{A}_i, i < n$:

$$\begin{aligned} A &\sqsubseteq I_s \sqcap \prod_{i < n} \bar{A}_i \\ I_s &\sqsubseteq \exists r. \top \sqcap \forall r. I_s \\ A_i \sqcap \prod_{j < i} A_j &\sqsubseteq \forall r. \bar{A}_i \\ \bar{A}_i \sqcap \prod_{j < i} A_j &\sqsubseteq \forall r. A_i \\ A_i \sqcap \bigsqcup_{j < i} \bar{A}_j &\sqsubseteq \forall r. A_i \\ \bar{A}_i \sqcap \bigsqcup_{j < i} \bar{A}_j &\sqsubseteq \forall r. \bar{A}_i \end{aligned}$$

Note that all points of the r -path satisfy a concept name I_s , from which we start the s -trees with two counters, realized using concept names U_i, \bar{U}_i and $V_i, \bar{V}_i, i < n$, and initialized to 0 and the value of the A -counter, respectively:

$$\begin{aligned} I_s &\sqsubseteq (U = 0) \\ I_s \sqcap A_j &\sqsubseteq V_j && j < n \\ I_s \sqcap \bar{A}_j &\sqsubseteq \bar{V}_j && j < n \\ \top &\sqsubseteq \exists s. \top \end{aligned}$$

Here, $(U = 0)$ is an abbreviation for the concept $\prod_{i=1}^n \bar{U}_i$, we use similar abbreviations below. The counters U_i and V_i are incremented along s analogously to how A_i is incremented along r , so we omit details. Configurations of M are represented between two consecutive points having U -counter value 0. We next enforce the structure of the computation tree, assuming that $q_0 \in Q_{\forall}$:

$$\begin{aligned} I_s &\sqsubseteq B_{\forall} \\ (U < 2^n - 1) \sqcap B_{\forall} &\sqsubseteq \forall s. B_{\forall} \\ (U < 2^n - 1) \sqcap B_{\exists}^i &\sqsubseteq \forall s. B_{\exists}^i && i \in \{1, 2\} \\ (U = 2^n - 1) \sqcap B_{\forall} &\sqsubseteq \forall s. (B_{\exists}^1 \sqcup B_{\exists}^2) \\ (U = 2^n - 1) \sqcap (B_{\exists}^1 \sqcup B_{\exists}^2) &\sqsubseteq \forall s. B_{\forall} \\ (U = 2^n - 1) \sqcap B_{\forall} &\sqsubseteq \exists s. Z \sqcap \exists s. \neg Z \end{aligned}$$

These sentences enforce that all points which represent a configuration satisfy exactly one of $B_{\forall}, B_{\exists}^1, B_{\exists}^2$ indicating the kind of configuration and, if existential, also a choice of the transition function. The symbol $Z \in \Sigma$ enforces the branching.

We next set the initial configuration, for input $w = a_0, \dots, a_{n-1}$.

$$\begin{aligned} A &\sqsubseteq A_{q_0, a_0} \\ A &\sqsubseteq \forall s^k . A_{a_k} && 0 < k < n \\ A &\sqsubseteq \forall s^{n+1} . \text{Blank} \\ \text{Blank} &\sqsubseteq A_{\square} \\ \text{Blank} \sqcap (U < 2^n - 1) &\sqsubseteq \forall s . \text{Blank} \end{aligned}$$

To coordinate consecutive configurations, we associate with M functions $f_i, i \in \{1, 2\}$ that map the content of three consecutive cells of a configuration to the content of the middle cell in the i -the successor configuration (assuming an arbitrary order on the set $\Delta(q, a)$). In what follows, we ignore the cornercases that occur at the border of configurations; they can be treated in a similar way. Clearly, for each possible such triple $(\sigma_1, \sigma_2, \sigma_3) \in \Gamma \cup (Q \times \Gamma)$, there is an \mathcal{ALC} concept $C_{\sigma_1, \sigma_2, \sigma_3}$ which is true at an element a of the computation tree iff a is labeled with A_{σ_1} , a 's s -successor b is labeled with A_{σ_2} , and b 's s -successor c is labeled with A_{σ_3} . Now, in each configuration, we synchronize elements with V -counter 0 by including for every $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ and $i \in \{1, 2\}$ the following sentences:

$$\begin{aligned} (V = 2^n - 1) \sqcap (U < 2^n - 2) \sqcap C_{\sigma_1, \sigma_2, \sigma_3} &\sqsubseteq \forall s . A_{f_1(\sigma)}^1 \sqcap \forall s . A_{f_2(\sigma)}^2 \\ (V = 2^n - 1) \sqcap (U < 2^n - 2) \sqcap C_{\sigma_1, \sigma_2, \sigma_3} \sqcap B_{\exists}^i &\sqsubseteq \forall s . A_{f_i(\sigma)}^i \end{aligned}$$

The concept names A_{σ}^i are used as markers (not in Σ) and are propagated along s for 2^n steps, exploiting the V -counter. The superscript $i \in \{1, 2\}$ determines the successor configuration that the symbol is referring to. After crossing the end of a configuration, the symbol σ is propagated using concept names A'_{σ} (the superscript is not needed anymore because the branching happens at the end of the configuration, based on Z).

$$\begin{aligned} (U < 2^n - 1) \sqcap A_{\sigma}^i &\sqsubseteq \forall s . A_{\sigma}^i \\ (U = 2^n - 1) \sqcap B_{\forall} \sqcap A_{\sigma}^1 &\sqsubseteq \forall s . (\neg Z \sqcup A'_{\sigma}) \\ (U = 2^n - 1) \sqcap B_{\forall} \sqcap A_{\sigma}^2 &\sqsubseteq \forall s . (Z \sqcup A'_{\sigma}) \\ (U = 2^n - 1) \sqcap B_{\exists}^i \sqcap A_{\sigma}^i &\sqsubseteq \forall s . A'_{\sigma} && i \in \{1, 2\} \\ (V < 2^n - 1) \sqcap A'_{\sigma} &\sqsubseteq \forall s . A'_{\sigma} \\ (V = 2^n - 1) \sqcap A'_{\sigma} &\sqsubseteq \forall s . A_{\sigma} \end{aligned}$$

For those (q, a) with $\Delta(q, a) = \emptyset$, we add the concept inclusion

$$A_{q, a} \sqsubseteq \perp.$$

The following Claim establishes correctness of the reduction

Claim. M accepts the input w iff there exists models $\mathfrak{A}, \mathfrak{B}$ of \mathcal{K} such that $\mathfrak{A}, a^{\mathfrak{A}} \sim_{\Sigma} \mathfrak{B}, b^{\mathfrak{B}}$.

Proof of the Claim. (\Rightarrow) If M accepts w , there is a computation tree of M on w . We construct a single interpretation \mathfrak{A} as follows. Let \mathfrak{A}^* be the infinite tree-shaped structure that represents the computation tree of M on w as described above, that is, configurations are represented by sequences of 2^n elements linked by role s and labeled by $B_{\forall}, B_{\exists}^1, B_{\exists}^2$ depending on whether the configuration is universal or existential, and in the latter case the superscript indicates which choice has been made for the existential state. Finally, the first element of the first successor configuration of a universal configuration is labeled with Z . Observe that \mathfrak{A}^* interprets only the symbols in Σ as non-empty. Now, we obtain structures $\mathfrak{A}_k, k < 2^n$ from \mathfrak{A}^* by interpreting non- Σ -symbols as follows:

- the root of \mathfrak{A}_k satisfies I_s ;
- the U -counter starts at 0 at the root and counts modulo 2^n along each s -path;
- the V -counter starts at k at the root and counts modulo 2^n along each s -path;
- the auxiliary concept names of the shape A_{σ}^i and A'_{σ} are interpreted in a minimal way so as to satisfy the concept inclusions listed above. Note that the respective concept inclusions are Horn, hence there is no choice.

Now obtain \mathfrak{A} from \mathfrak{A}^* and the \mathfrak{A}_k by creating an (both side) infinite r -path ρ through $a^{\mathfrak{A}} = a$ (with the corresponding A -counter) and adding all \mathfrak{A}_k to every node on the r -path by identifying the roots of the \mathfrak{A}_k with the node on the path. Additionally, add \mathfrak{A}^* to $b^{\mathfrak{A}} = b$ by identifying b with the root of \mathfrak{A}^* . It should be clear that \mathfrak{A} is as required. In particular, \mathfrak{A} is a model of \mathcal{O} and the reflexive and symmetric closure of

- all pairs $(b, e), (e, e')$, with e, e' on ρ , and
- all pairs $(e, e'), (e', e'')$, with e in \mathfrak{A}^* and e', e'' copies of e in the trees \mathfrak{A}_k .

is an $\mathcal{ALCT}(\Sigma)$ -bisimulation S on \mathfrak{A} with $(b, a) \in S$.

(\Leftarrow) Let $\mathfrak{A}, \mathfrak{B}$ be models of \mathcal{K} such that $\mathfrak{A}, a^{\mathfrak{A}} \sim_{\Sigma} \mathfrak{B}, b^{\mathfrak{B}}$. As it was argued above, due to the r -self loop at $b^{\mathfrak{A}}$, from $a^{\mathfrak{A}}$ there has to be an outgoing infinite r -path on which all s -trees are Σ -bisimilar. There is also an outgoing infinite r^- -path with this property, but it is not relevant for the proof. All those s -trees are additionally labeled with some auxiliary concept names not in Σ , depending on the distance from $a^{\mathfrak{A}}$. However, it can be shown using the CIs in \mathcal{O} that all s -trees contain a computation tree of M on input w .

Note that we have to take care of inverses in the correctness proof since the characterization refers to \mathcal{ALCT} -bisimulations. Since the ontology is actually an \mathcal{ALC} -ontology (and there is a similar characterization), also strong separability in \mathcal{ALC} is 2EXPTIME-hard.

8 Strong Separability in GF and GNF

We show that strong separability with signature is decidable in the guarded fragment, GF, and the guarded negation fragment, GNF, of FO. We also obtain a 3EXPTIME upper bound for GF and 2EXPTIME-completeness for GNF. Finally, we show that strong GNF separability with signature restrictions coincides with strong FO separability with signature restrictions for labeled GNF-KBs. The analogous result does not hold for GF. The proofs are based on the link to interpolants and the CIP discussed in Section 6.

The formulas $\varphi_{\Sigma, \mathbf{a}}(\mathbf{x})$ and $\neg\varphi_{\Sigma, \mathbf{b}}(\mathbf{x})$ defined in Section 6 are not in GF nor GNF, even if \mathcal{K} is a KB in GF or, respectively, GNF. To obtain formulas in GF and GNF, take fresh relation symbols $R_{\mathcal{D}, \mathbf{a}}$ and $R_{\mathcal{D}, \mathbf{b}}$ of arity n , where n is the number of constants in \mathcal{D} . Then add $R_{\mathcal{D}, \mathbf{a}}(\mathbf{y})$ to $\mathcal{K}_{\Sigma, \mathbf{a}}$ when constructing $\varphi_{\Sigma, \mathbf{a}}(\mathbf{x})$, where \mathbf{y} is an enumeration of the variables in $\mathcal{K}_{\Sigma, \mathbf{a}}$. Denote the resulting formula by $\varphi'_{\Sigma, \mathbf{a}}(\mathbf{x})$. Do the same to construct $\varphi'_{\Sigma, \mathbf{b}}(\mathbf{x})$, using $R_{\mathcal{D}, \mathbf{b}}$ instead of $R_{\mathcal{D}, \mathbf{a}}$. The formulas $\varphi'_{\Sigma, \mathbf{a}}$ and $\neg\varphi'_{\Sigma, \mathbf{b}}$ are in GF and GNF if the KB is given in GF and, respectively, GNF. By construction we obtain the following result.

Theorem 8. *Let $\mathcal{L} \in \{GF, GNF\}$. Then there is a polynomial time reduction of strong \mathcal{L} -separability with signature to \mathcal{L} -interpolant existence. Moreover, given a labeled \mathcal{L} -KB $(\mathcal{K}, \{\mathbf{a}\}, \{\mathbf{b}\})$ and $\Sigma \subseteq \text{sig}(\mathcal{K})$, the following conditions are equivalent for any formula φ in \mathcal{L} :*

1. φ strongly $\mathcal{L}(\Sigma)$ -separates $(\mathcal{K}, \{\mathbf{a}\}, \{\mathbf{b}\})$;
2. φ is an \mathcal{L} -interpolant for $\varphi'_{\Sigma, \mathbf{a}}(\mathbf{x}), \neg\varphi'_{\Sigma, \mathbf{b}}(\mathbf{x})$.

It has been proved in [10, 8] that GNF has the CIP. Thus, we obtain the following result.

Theorem 9. *Strong GNF-separability with signature is 2EXPTIME-complete. Moreover, a GNF-KB $(\mathcal{K}, \{\mathbf{a}\}, \{\mathbf{b}\})$ is strongly GNF(Σ)-separable iff it is strongly FO(Σ)-separable.*

In contrast, GF does not enjoy the CIP [24] and so interpolant existence in GF does not reduce to a validity. In fact, decidability and a 3EXPTIME upper bound for GF-interpolant existence has only recently been established [28]. Thus, we obtain the following upper bound.

Theorem 10. *Strong GF-separability with signature is in 3EXPTIME.*

We conjecture that the 3EXPTIME complexity bound for GF is tight. As GF does not enjoy the CIP, we also do not obtain that a GF-KB $(\mathcal{K}, \{\mathbf{a}\}, \{\mathbf{b}\})$ is strongly GF(Σ)-separable iff it is strongly FO(Σ)-separable. In fact, the counterexample to CIP constructed in [8] is easily adapted to show the following.

Theorem 11. *Strong FO(Σ)-separability of a GF-KB $(\mathcal{K}, \{\mathbf{a}\}, \{\mathbf{b}\})$ does not imply strong GF(Σ)-separability of $(\mathcal{K}, \{\mathbf{a}\}, \{\mathbf{b}\})$.*

9 Conclusion

We have investigated the complexity of deciding weak and strong separability of labeled KBs with signature restrictions for \mathcal{ALCT} and guarded fragments of FO, and observed a close link between weak separability and uniform interpolants on the one hand, and between strong separability and Craig interpolants on the other. Numerous questions remain to be explored: what is the size of separating formulas and how can they be computed efficiently, if they exist? What is the complexity of weak non-projective separability with signature restrictions for \mathcal{ALCT} ? We conjecture that this is still 2EXPTIME -complete but lack a proof. What happens for DLs with number restrictions and/or nominals? We have shown that weak projective separability is undecidable for \mathcal{ALCFIO} with signature restrictions, but it could well be decidable for \mathcal{ALCQO} . For strong separability, there are many exciting open problems: is strong separability with signature restrictions decidable for \mathcal{ALCFIO} ? In this case, even the case without signature restrictions has not yet been investigated and could well already be tricky. Is it decidable for the two-variable fragment of FO? For the two-variable fragment, the case without signature restrictions has been investigated in [26], and NEXPTIME -completeness established. Attacking these problems is closely related to deciding the existence of Craig interpolants and computing (good) separating formulas is closely related to computing (good) Craig interpolants. Also of interest are the same questions for Horn DLs. The situation for \mathcal{EL} and \mathcal{ELI} has been explored in [18, 27], but more expressive ones have not yet been considered.

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A Proof of Theorem 1

We formulate the result to be shown again.

Theorem 1 Assume a labeled \mathcal{ALCI} -KB $(\mathcal{K}, P, \{b\})$ and $\Sigma \subseteq \text{sig}(\mathcal{K})$ are given. Then the following conditions are equivalent:

1. $\mathcal{K} = (\mathcal{O}, \mathcal{D})$ is projectively $\mathcal{ALCI}(\Sigma)$ -separable.
2. there exists a forest model \mathfrak{A} of \mathcal{K} of finite outdegree and a signature Σ' such that $\Sigma' \cap \text{sig}(\mathcal{K}) \subseteq \Sigma$ and for all models \mathfrak{B} of \mathcal{K} and all $a \in P$: $\mathfrak{B}, a^{\mathfrak{B}} \not\sim_{\mathcal{ALCI}, \Sigma'} \mathfrak{A}, b^{\mathfrak{A}}$.
3. there exists a forest model \mathfrak{A} of \mathcal{K} of finite outdegree such that for all models \mathfrak{B} of \mathcal{K} and all $a \in P$: $\mathfrak{B}, a^{\mathfrak{B}} \not\sim_{\mathcal{ALCI}, \Sigma}^f \mathfrak{A}, b^{\mathfrak{A}}$.
4. there exists a forest model \mathfrak{A} of \mathcal{K} of finite outdegree such that for all $a \in P$: $\mathcal{D}_{\text{con}(a)}, a \not\sim_c^\Sigma \mathfrak{A}, b^{\mathfrak{A}}$.

Proof. “1 \Rightarrow 2”. Take an \mathcal{ALCI} -concept C with $\text{sig}(C) \cap \text{sig}(\mathcal{K}) \subseteq \Sigma$ such C separates $(\mathcal{K}, P, \{b\})$. There exists a model \mathfrak{A} of \mathcal{K} of finite outdegree such that $b^{\mathfrak{A}} \in (-C)^{\mathfrak{A}}$. Let $\Sigma' = \text{sig}(C)$. Then \mathfrak{A} and Σ' are as required for Condition 2.

“2 \Rightarrow 3”. Take a forest model \mathfrak{A} and Σ' such that Condition 2 holds. We claim that Condition 3 holds for \mathfrak{A} as well. Suppose that there exists a model \mathfrak{B} of \mathcal{K} , $a \in P$, and a functional Σ -bisimulation f witnessing $\mathfrak{B}, a^{\mathfrak{B}} \sim_{\mathcal{ALCI}, \Sigma}^f \mathfrak{A}, b^{\mathfrak{A}}$.

Define \mathfrak{B}' by expanding \mathfrak{B} as follows:

- for every concept name $A \in \Sigma' \setminus \text{sig}(\mathcal{K})$ and $d \in \text{dom}(f)$, let $d \in A^{\mathfrak{B}'}$ if $f(d) \in A^{\mathfrak{A}}$;
- for every role R over $\Sigma' \setminus \text{sig}(\mathcal{K})$ and $d \in \text{dom}(f)$, if there exists $e \in \text{dom}(\mathfrak{A})$ with $(f(d), e) \in R^{\mathfrak{A}}$, then add a disjoint copy of \mathfrak{A} to \mathfrak{B} and add (d, e') to $R^{\mathfrak{B}'}$ for the copy e' of e .

It is easy to see that $\mathfrak{B}', a^{\mathfrak{B}'} \sim_{\mathcal{ALCI}, \Sigma'} \mathfrak{A}, b^{\mathfrak{A}}$, and we have derived a contradiction.

“3 \Rightarrow 4”. Take a forest model \mathfrak{A} such that Condition 3 holds. We claim that Condition 4 holds for \mathfrak{A} as well. For a proof by contradiction let h be a Σ -homomorphism and t_d , $d \in \text{dom}(\mathcal{D})$, be \mathcal{K} -types, and $a \in P$ such h refutes Condition 4. Take models \mathfrak{B}_d of \mathcal{O} such that $\mathfrak{B}_d, d \sim_{\mathcal{ALCI}, \Sigma} \mathfrak{A}, h(d)$. We may assume that the \mathfrak{B}_d are tree-shaped with root d and that the bisimulations are functions f_d . Now attach to every $d \in \text{dom}(\mathcal{D})$ the model \mathfrak{B}_d and obtain \mathfrak{B} by adding (d, d') to $R^{\mathfrak{B}}$ if $R(d, d') \in \mathcal{D}$. Then

$$f = \bigcup_{d \in \text{dom}(\mathcal{D})} f_d$$

is a functional $\mathcal{ALCI}(\Sigma)$ -bisimulation between \mathfrak{B} and \mathfrak{A} .

“4 \Rightarrow 3”. Take a forest model \mathfrak{A} such that Condition 4 holds. We claim that Condition 3 holds for \mathfrak{A} as well. For a proof by contradiction let f be a functional $\mathcal{ALCI}(\Sigma)$ -bisimulation witnessing $\mathfrak{B}, a^{\mathfrak{B}} \sim_{\mathcal{ALCI}, \Sigma}^f \mathfrak{A}, b^{\mathfrak{A}}$ for some model \mathfrak{B} of \mathcal{K} .

The restriction h of f of \mathcal{D} is the Σ -homomorphism needed to refute Condition 4.

“3 \Rightarrow 2”. Take a model \mathfrak{A} such that Condition 3 holds. Define \mathfrak{A}' by expanding \mathfrak{A} as follows. Take for any $d \in \text{dom}(\mathfrak{A})$ a fresh concept name A_d and set $A_d^{\mathfrak{A}'} = \{d\}$. Clearly Condition 2 holds for \mathfrak{A}' and $\Sigma' = \Sigma \cup \{A_d \mid d \in \text{dom}(\mathfrak{A})\}$.

“2 \Rightarrow 1”. Straightforward. \square

A.1 Additional Definitions for 2ATAs

We make precise the semantics of 2ATAs. Let $\mathcal{A} = (Q, \Theta, q_0, \delta, \Omega)$ be a 2ATA and (T, L) a Θ -labeled tree. A *run for \mathcal{A} on (T, L)* is a $T \times Q$ -labeled tree (T_r, r) such that:

- $\varepsilon \in T_r$ and $r(\varepsilon) = (\varepsilon, q_0)$;
- For all $y \in T_r$ with $r(y) = (x, q)$ and $\delta(q, L(x)) = \varphi$, there is an assignment v of truth values to the transitions in φ such that v satisfies φ and:
 - if $v(p) = 1$, then $r(y') = (x, p)$ for some successor y' of y in T_r ;
 - if $v(\langle - \rangle p) = 1$, then $x \neq \varepsilon$ and there is a successor y' of y in T_r with $r(y') = (x \cdot -1, p)$;
 - if $v([-]p) = 1$, then $x = \varepsilon$ or there is a successor y' of y in T_r such that $r(y') = (x \cdot -1, p)$;
 - if $v(\diamond p) = 1$, then there is a successor x' of x in T and a successor y' of y in T_r such that $r(y') = (x', p)$;
 - if $v(\square p) = 1$, then for every successor x' of x in T , there is a successor y' of y in T_r such that $r(y') = (x', p)$.

Let $\gamma = i_0 i_1 \dots$ be an infinite path in T_r and denote, for all $j \geq 0$, with q_j the state such that $r(i_0 \dots i_j) = (x, q_j)$. The path γ is *accepting* if the largest number m such that $\Omega(q_j) = m$ for infinitely many j is even. A run (T_r, r) is accepting, if all infinite paths in T_r are accepting. Finally, a tree is accepted if there is some accepting run for it.