$\mathcal{EL}$-Concepts go Second-Order: Greatest Fixpoints and Simulation Quantifiers

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1 Introduction

The well-known description logic (DL) $\mathcal{ALC}$ is usually regarded as the *basic DL* that comprises all Boolean concept constructors from which all expressive DLs are derived by admitting additional concept constructors. The fundamental role of $\mathcal{ALC}$ is largely due to the fact that it is very well-behaved regarding its logical, model-theoretic, and computational properties. This good behavior can, in turn, be explained nicely by the fact that $\mathcal{ALC}$-concepts comprise exactly the bisimulation invariant fragment of first-order logic (FO): an FO formula is invariant under bisimulation if, and only if, it is equivalent to an $\mathcal{ALC}$-concept [21, 12, 15]. For example, invariance under bisimulation can explain the tree-model property of $\mathcal{ALC}$ as well as its favorable computational properties [23]. In this characterization, the condition that $\mathcal{ALC}$ is a fragment of FO is much less important than its bisimulation invariance. In fact, $\mathcal{ALC}\mu$, the extension of $\mathcal{ALC}$ with fixpoint operators, is not a fragment of FO, but inherits almost all important properties of $\mathcal{ALC}$ [7, 11]. Similar to $\mathcal{ALC}$, $\mathcal{ALC}\mu$'s fundamental role (in particular in its formulation as the modal mu-calculus) can be explained by the fact that $\mathcal{ALC}\mu$-concepts comprise exactly the bisimulation invariant fragment of monadic second-order logic (MSO) [13, 7]. Indeed, from a purely theoretical viewpoint it is hard to explain why $\mathcal{ALC}$ rather than $\mathcal{ALC}\mu$ forms the logical underpinning of current ontology language standards; the current state of the art in automated reasoning and the fact that mu-calculus concepts can be hard to grasp are probably the only reasons for the limited interest in $\mathcal{ALC}\mu$ compared to $\mathcal{ALC}$.

In recent years, the development of very large ontologies and the use of ontologies to access instance data has led to a revival of interest in *tractable* DLs. The main examples are $\mathcal{EL}$ [4] and DL-Lite [8], the logical underpinnings of the OWL profiles OWL2 EL and OWL2 QL, respectively. In contrast to $\mathcal{ALC}$, a satisfactory characterization of the expressivity of such DLs is still missing, and a first aim of this paper is to fill this gap for $\mathcal{EL}$. To this end, we characterize $\mathcal{EL}$ as a maximal fragment of FO that is preserved under *simulations* and has *finite minimal models*. Note that preservation under simulations alone would characterize $\mathcal{EL}$ with disjunctions, and the existence of minimal models reflects the "Horn-aspect" of $\mathcal{EL}$.

The second and main aim of this paper, however, is to introduce and investigate two equi-expressive extensions of $\mathcal{EL}$ with greatest fixpoints, $\mathcal{EL}^{\nu}$ and $\mathcal{EL}^{\nu+}$, and to prove that they stand in a similar relationship to $\mathcal{EL}$ as $\mathcal{ALC}\mu$ to $\mathcal{ALC}$. To this end, we prove that $\mathcal{EL}^\nu$ (and therefore also $\mathcal{EL}^{\nu+}$, which admits mutual fixpoints and is exponentially more succinct than $\mathcal{EL}^\nu$) can be characterized as a maximal fragment of MSO that is preserved under *simulations* and has *finite minimal models*. Similar to $\mathcal{ALC}\mu$, $\mathcal{EL}^\nu$ and $\mathcal{EL}^{\nu+}$ inherit many good properties of $\mathcal{EL}$, the most interesting being that *reasoning with general concept inclusions (GCIs) is still tractable*. Thus, in contrast to $\mathcal{ALC}\mu$, the development of practical decision procedures is no obstacle to using $\mathcal{EL}^\nu$. Moreover, $\mathcal{EL}^{\nu+}$ has a number of rather useful properties that $\mathcal{EL}$ and most of its extensions are lacking. In this paper, we give two examples: in $\mathcal{EL}^{\nu+}$ *most specific concepts* always exist, are of polynomial size, and can be computed in polytime; and $\mathcal{EL}^{\nu+}$ has the *Beth definability property* with explicit definitions being computable in polytime and of polynomial size. Another application of $\mathcal{EL}^{\nu+}$ is demonstrated in [14], where the succinct representations of definitions in $\mathcal{EL}^{\nu+}$ are used to develop polytime algorithms for decomposing general $\mathcal{EL}$-TBoxes. To prove these result and provide a better understanding of the modeling capabilities of $\mathcal{EL}^{\nu+}$ we show that it has the same expressive power as extensions of $\mathcal{EL}$ by means of *simulation quantifiers*, a variant of second-order quantifiers that quantifies "modulo a simulation of the model"; in fact, the relationship between simulation quantifiers and $\mathcal{EL}^{\nu+}$ is somewhat similar to the relationship between $\mathcal{ALC}\mu$ and bisimulation quantifiers [10]. Most proofs are given in the appendix.

2 Preliminaries

Let $\mathsf{N_C}$ and $\mathsf{N_R}$ be countably infinite and mutually disjoint sets of concept and role names. *$\mathcal{EL}$-concepts* are built according to the rule

$$C := \quad A \quad | \quad \top \quad | \quad \bot \quad | \quad C \sqcap D \quad | \quad \exists r.C,$$

where $A \in \mathsf{N_C}$, $r \in \mathsf{N_R}$, and C, D range over $\mathcal{EL}$-concepts[3]. An *$\mathcal{EL}$-concept inclusion* takes the form $C \sqsubseteq D$, where C, D are $\mathcal{EL}$-concepts. A *general $\mathcal{EL}$-TBox $\mathcal{T}$* is a finite set of $\mathcal{EL}$-concept inclusions. An *ABox assertion* is an expression of the form $A(a)$ or $r(a, b)$, where a, b are from a countably infinite set of individual names $\mathsf{N_I}$, $A \in \mathsf{N_C}$, and $r \in \mathsf{N_R}$. An *ABox* is a finite set of ABox assertions. By $\mathsf{Ind}(\mathcal{A})$ we denote the set of individual names in $\mathcal{A}$. An $\mathcal{EL}$-knowledge base (KB) is a pair $(\mathcal{T}, \mathcal{A})$ that consists of an $\mathcal{EL}$-TBox $\mathcal{T}$ and an ABox $\mathcal{A}$.

The semantics of $\mathcal{EL}$ is based on interpretations $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$, where the *domain* $\Delta^\mathcal{I}$ is a non-empty set, and $\cdot^\mathcal{I}$ is a function mapping each concept name A to a subset $A^\mathcal{I}$ of $\Delta^\mathcal{I}$, each role name r to a binary relation $r^\mathcal{I} \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I}$, and each individual name a to an element $a^\mathcal{I}$ of $\Delta^\mathcal{I}$. The interpretation $C^\mathcal{I} \subseteq \Delta^\mathcal{I}$ of $\mathcal{EL}$-concepts C in an interpretation $\mathcal{I}$ is defined in the standard way [5]. We will often make use of the fact that $\mathcal{EL}$-concepts can be regarded as formulas in FO (and, therefore, MSO) with unary predicates from $\mathsf{N_C}$, binary predicates from $\mathsf{N_R}$, and exactly one free variable [5]. We will often not distinguish between $\mathcal{EL}$-concepts and their translations into FO/MSO.

We now introduce $\mathcal{EL}^\nu$, the extension of $\mathcal{EL}$ with greatest fixpoints and the main language studied in this paper. *$\mathcal{EL}^\nu$-concepts* are defined like $\mathcal{EL}$-concepts, but additionally allow the greatest fixpoint constructor $\nu X.C$, where X is from a countably

[3] In the literature, $\mathcal{EL}$ is typically defined without $\bot$. The sole purpose of including $\bot$ here, is to simplify the formulation of some results.

infinite set of *(concept) variables* $\mathsf{N_V}$ and C an $\mathcal{EL}^\nu$-concept. A variable is *free* in a concept C if it occurs in C at least once outside the scope of any ν-constructor that binds it. An $\mathcal{EL}^\nu$-concept is *closed* if it does not contain any free variables. An $\mathcal{EL}^\nu$-*concept inclusion* takes the form $C \sqsubseteq D$, where C, D are are closed $\mathcal{EL}^\nu$-concepts. The semantics of the greatest fixpoint constructor is as follows, where $\mathcal{V}$ is an *assignment* that maps variables to subsets of $\Delta^\mathcal{I}$ and $\mathcal{V}[X \mapsto W]$ denotes $\mathcal{V}$ modified by setting $\mathcal{V}(X) = W$:

$$(\nu X.C)^{\mathcal{I},\mathcal{V}} = \bigcup\{W \subseteq \Delta^\mathcal{I} \mid W \subseteq C^{\mathcal{I},\mathcal{V}[X\mapsto W]}\}$$

We will also consider an extended version of the ν-constructor that allows to capture mutual recursion. It has been considered e.g. in [9, 22] and used in a DL context in [18]; it can be seen as a variation of the fixpoint equations considered in [7]. The constructor has the form $\nu_i X_1 \cdots X_n.C_1, \ldots, C_n$ where $1 \leq i \leq n$. The semantics is defined by setting $(\nu_i X_1 \cdots X_n.C_1, \ldots, C_n)^{\mathcal{I},\mathcal{V}}$ to

$$\bigcup\{W_i \mid \exists W_1, \ldots, W_{i-1}, W_{i+1}, \ldots, W_n \text{ s.t. for } 1 \leq j \leq n: \\ W_j \subseteq C_j^{\mathcal{I},\mathcal{V}[X_1\mapsto W_1,\ldots,X_n\mapsto W_n]}\}$$

We use $\mathcal{EL}^{\nu+}$ to denote $\mathcal{EL}$ extended with this mutual greatest fixpoint constructor. Clearly, $\nu X.C \equiv \nu_1 X.C$, thus every $\mathcal{EL}^\nu$-concept is equivalent to an $\mathcal{EL}^{\nu+}$-concept. Conversely, we have the following result [7]:

Proposition 1. *For every $\mathcal{EL}^{\nu+}$-concept, one can construct an equivalent $\mathcal{EL}^\nu$-concept of at most exponential size.*

By extending the translation of $\mathcal{EL}$-concepts into FO in the obvious way, one can translate closed $\mathcal{EL}^{\nu+}$-concepts into a MSO formula with one free first-order variable. We will often not distinguish between $\mathcal{EL}^{\nu+}$-concepts and their translation into MSO.

3 Characterizing $\mathcal{EL}$ using simulations

The purpose of this section is to provide a model-theoretic characterization of $\mathcal{EL}$ as a fragment of FO that is similar in spirit to the well-known characterization of $\mathcal{ALC}$ as the bisimulation-invariant fragment of FO. To this end, we first characterize $\mathcal{EL}^\sqcup$, the extension of $\mathcal{EL}$ with the disjunction constructor $\sqcup$, as the fragment of FO that is preserved under simulation. Then we characterize the fragment $\mathcal{EL}$ of $\mathcal{EL}^\sqcup$ using, in addition, the existence of minimal models. A *pointed interpretation* is a pair $(\mathcal{I}, d)$ consisting of an interpretation $\mathcal{I}$ and $d \in \Delta^\mathcal{I}$. A *signature* Σ is a set of concept and role names.

Definition 1 (Simulations). Let $(\mathcal{I}_1, d_1)$ and $(\mathcal{I}_2, d_2)$ be pointed interpretations and Σ a signature. A relation $S \subseteq \Delta^{\mathcal{I}_1} \times \Delta^{\mathcal{I}_2}$ is a Σ-*simulation* between $(\mathcal{I}_1, d_1)$ and $(\mathcal{I}_2, d_2)$, in symbols $S : (\mathcal{I}_1, d_1) \leq_\Sigma (\mathcal{I}_2, d_2)$, if $(d_1, d_2) \in S$ and the following conditions hold:

1. for all concept names $A \in \Sigma$ and all $(e_1, e_2) \in S$, if $e_1 \in A^{\mathcal{I}_1}$ then $e_2 \in A^{\mathcal{I}_2}$;
2. for all role names $r \in \Sigma$, all $(e_1, e_2) \in S$, and all $e_1' \in \Delta^{\mathcal{I}_1}$ with $(e_1, e_1') \in r^{\mathcal{I}_1}$, there exists $e_2' \in \Delta^{\mathcal{I}_2}$ such that $(e_2, e_2') \in r^{\mathcal{I}_2}$ and $(e_1', e_2') \in S$.

If such an S exists, then we also say that $(\mathcal{I}_2, d_2)$ *Σ-simulates* $(\mathcal{I}_1, d_1)$ and write $(\mathcal{I}_1, d_1) \leq_\Sigma (\mathcal{I}_2, d_2)$.

If $\Sigma = \mathsf{N_C} \cup \mathsf{N_R}$, then we omit Σ and use the term *simulation* to denote Σ-simulations and $(\mathcal{I}_1, d_1) \leq (\mathcal{I}_2, d_2)$ stands for $(\mathcal{I}_1, d_1) \leq_\Sigma (\mathcal{I}_2, d_2)$. It is well-known that the description logic $\mathcal{EL}$ is intimately related to the notion of a simulation, see for example [3, 16]. In particular, $\mathcal{EL}$-concepts are preserved under simulations in the sense that if $d \in C^\mathcal{I}$ for an $\mathcal{EL}$-concept C and $(\mathcal{I}_1, d_1) \leq_\Sigma (\mathcal{I}_2, d_2)$, then $d_2 \in C^{\mathcal{I}_2}$. This observation, which clearly generalizes to $\mathcal{EL}^\sqcup$, illustrates the (limitations of the) modeling capabilities of $\mathcal{EL}$/$\mathcal{EL}^\sqcup$. We now strengthen it to an exact characterization of the expressive power of these logics relative to FO.

Let $\varphi(x)$ be an FO-formula (or, later, MSO-formula) with one free variable x. We say that $\varphi(x)$ is *preserved under simulations* if, and only if, for all $(\mathcal{I}_1, d_1)$ and $(\mathcal{I}_2, d_2)$, $\mathcal{I}_1 \models \varphi[d_1]$ and $(\mathcal{I}_1, d_1) \leq (\mathcal{I}_2, d_2)$ implies $\mathcal{I}_2 \models \varphi[d_2]$.

Theorem 1. *An FO-formula $\varphi(x)$ is preserved under simulations if, and only if, it is equivalent to an $\mathcal{EL}^\sqcup$-concept.*

To characterize $\mathcal{EL}$, we add a central property of Horn-logics on top of preservation under simulations. Let $\mathcal{L}$ be a set of FO (or, later, MSO) formulas, each with one free variable. We say that $\mathcal{L}$ *has (finite) minimal models* if, and only if, for every $\varphi(x) \in \mathcal{L}$ there exists a (finite) pointed interpretation $(\mathcal{I}, d)$ such that for all $\psi(x) \in \mathcal{L}$, we have $\mathcal{I} \models \psi[d]$ if, and only if, $\forall x.(\varphi(x) \rightarrow \psi(x))$ is a tautology.

Theorem 2. *The set of $\mathcal{EL}$-concepts is a maximal set of FO-formulas that is preserved under simulations and has minimal models (equivalently: has finite minimal models): if $\mathcal{L}$ is a set of FO-formulas that properly contains all $\mathcal{EL}$-concepts, then either it contains a formula not preserved under simulations or it does not have (finite) minimal models.*

We note that de Rijke and Kurtonina have given similar characterizations of various non-Boolean fragments of $\mathcal{ALC}$. In particular, Theorem 1 is rather closely related to results proved in [15] and would certainly have been included in the extensive list of characterizations given there had $\mathcal{EL}$ already been as popular as it is today. In contrast, the novelty of Theorem 2 is that it makes the Horn character of $\mathcal{EL}$ explicit through minimal models while the characterizations of disjunction-free languages in [15] are based on simulations that take sets (rather than domain-elements) as arguments.

4 Simulation quantifiers and $\mathcal{EL}^\nu$

To understand and characterize the expressive power and modeling capabilities of $\mathcal{EL}^\nu$, we introduce three distinct types of simulation quantifiers and show that, in each case, the resulting language has the same expressive power as $\mathcal{EL}^\nu$.

Simulating interpretations. The first language $\mathcal{EL}^{si}$ extends $\mathcal{EL}$ by the concept constructor $\exists^{sim}(\mathcal{I}, d)$, where $(\mathcal{I}, d)$ is a finite pointed interpretation in which only finitely many $\sigma \in \mathsf{N_C} \cup \mathsf{N_R}$ have a non-empty interpretation $\sigma^\mathcal{I} \subseteq \Delta^\mathcal{I}$. The semantics of $\exists^{sim}(\mathcal{I}, d)$ is defined by setting for all interpretations $\mathcal{J}$ and $e \in \Delta^\mathcal{J}$,

$$e \in (\exists^{sim}(\mathcal{I}, d))^\mathcal{J} \text{ iff } (\mathcal{I}, d) \leq (\mathcal{J}, e).$$

Example 1. Let $\mathcal{I}$ consist of one point d such that $(d,d) \in r^{\mathcal{I}}$. Then $e \in (\exists^{sim}(\mathcal{I},d))^{\mathcal{J}}$ iff there is an infinite r-chain starting at e in $\mathcal{I}$, i.e., there exist $e_0, e_1, e_2, \dots$ such that $e = e_0$ and $(e_i, e_{i+1}) \in r^{\mathcal{J}}$ for all $i \geq 0$.

To attain a better understanding of the constructor $\exists^{sim}$, it is interesting to observe that *every* $\mathcal{EL}^{si}$-concept is equivalent to a concept of the form $\exists^{sim}(\mathcal{I},d)$.

Lemma 1. *For every $\mathcal{EL}^{si}$-concept C one can construct, in linear time, an equivalent concept of the form $\exists^{sim}(\mathcal{I},d)$.*

Proof. By induction on the construction of C. If $C = A$ for a concept name A, then let $\mathcal{I} = (\{d\}, \cdot^{\mathcal{I}})$, where $A^{\mathcal{I}} = \{d\}$ and $\sigma^{\mathcal{I}} = \emptyset$ for all symbols distinct from A. Clearly, A and $\exists^{sim}(\mathcal{I},d)$ are equivalent. For $C_1 = \exists^{sim}(\mathcal{I}_1, d_1)$ and $C_2 = \exists^{sim}(\mathcal{I}_2, d_2)$ assume that $\Delta^{\mathcal{I}_1} \cap \Delta^{\mathcal{I}_2} = \{d_1\} = \{d_2\}$. Then $\exists^{sim}(\mathcal{I}_1 \cup \mathcal{I}_2, d_1)$ is equivalent to $C_1 \sqcap C_2$. For $C = \exists r.\exists^{sim}(\mathcal{I},d)$ construct a new interpretation $\mathcal{I}'$ by adding a new node e to $\Delta^{\mathcal{I}}$ and setting $(e,d) \in r^{\mathcal{I}'}$. Then $\exists^{sim}(\mathcal{I}',e)$ and C are equivalent. □

We will show that there are polynomial translations between $\mathcal{EL}^{si}$ and $\mathcal{EL}^{\nu+}$. When using $\mathcal{EL}^{\nu}$ in applications and to provide a translation from $\mathcal{EL}^{\nu+}$ to $\mathcal{EL}^{si}$, it is convenient to have available a "syntactic" simulation operator.

Simulating models of TBoxes. The second language $\mathcal{EL}^{st}$ extends $\mathcal{EL}$ by the concept constructor $\exists^{sim}\Sigma.(\mathcal{T},C)$, where Σ is a finite signature, $\mathcal{T}$ a general TBox, and C a concept. To admit nestings of $\exists^{sim}$, the concepts of $\mathcal{EL}^{st}$ are defined by simultaneous induction; namely, $\mathcal{EL}^{st}$-concepts, concept inclusions, and general TBoxes are defined as follows:

- every $\mathcal{EL}$-concept, concept inclusion, and general TBox is an $\mathcal{EL}^{st}$-concept, concept inclusion, and general TBox, respectively;
- if $\mathcal{T}$ is a general $\mathcal{EL}^{st}$-TBox, C an $\mathcal{EL}^{st}$-concept, and Σ a finite signature, then $\exists^{sim}\Sigma.(\mathcal{T},C)$ is an $\mathcal{EL}^{st}$-concept;
- if C, D are $\mathcal{EL}^{st}$-concepts, then $C \sqsubseteq D$ is a $\mathcal{EL}^{st}$-concept inclusion;
- a general $\mathcal{EL}^{st}$-TBox is a finite set of $\mathcal{EL}^{st}$-concept inclusions.

The semantics of $\exists^{sim}\Sigma.(\mathcal{T},C)$ is as follows:

> $d \in (\exists^{sim}\Sigma.(\mathcal{T},C))^{\mathcal{I}}$ iff there exists $(\mathcal{J},e)$ such that $\mathcal{J}$ is a model of $\mathcal{T}$, $e \in C^{\mathcal{J}}$ and $(\mathcal{J},e) \leq_{\Gamma} (\mathcal{I},d)$, where $\Gamma = (\mathsf{N_C} \cup \mathsf{N_R}) \setminus \Sigma$.

Example 2. Let $\mathcal{T} = \{A \sqsubseteq \exists r.A\}$ and $\Sigma = \{A\}$. Then $\exists^{sim}\Sigma.(\mathcal{T},A)$ is equivalent to the concept $\exists^{sim}(\mathcal{I},d)$ defined in Example 1.

We will later exploit the fact that $\exists^{sim}\Sigma.(\mathcal{T},C)$ is equivalent to $\exists^{sim}\Sigma \cup \{A\}.(\mathcal{T}',A)$, where A is a fresh concept name and $\mathcal{T}' = \mathcal{T} \cup \{A \sqsubseteq C\}$. Another interesting (but subsequently unexploited) observation is that we can w.l.o.g. restrict Σ to singleton sets since

$$\exists^{sim}(\{\sigma\} \cup \Sigma).(\mathcal{T},C) \equiv \exists^{sim}\{\sigma\}.(\emptyset, \exists^{sim}\Sigma.(\mathcal{T},C))$$
$$\exists^{sim}\emptyset.(\mathcal{T},C) \equiv \exists^{sim}\{B\}.(\mathcal{T},C)$$

where B is a concept name that does not occur in $\mathcal{T}$ and C.

Simulating models of KBs. The third language $\mathcal{EL}^{sa}$ extends $\mathcal{EL}$ by the concept constructor $\exists^{sim}\Sigma.(\mathcal{T}, \mathcal{A}, a)$, where a is an individual name in the ABox $\mathcal{A}$, $\mathcal{T}$ is a TBox, and Σ a finite signature. More precisely, we define $\mathcal{EL}^{sa}$-concepts, concept inclusions, general TBoxes, and KBs, by simultaneous induction as follows:

- every $\mathcal{EL}$-concept, concept inclusion, general TBox, and KB is an $\mathcal{EL}^{sa}$-concept, concept inclusion, general TBox, and KB, respectively;
- if $(\mathcal{T}, \mathcal{A})$ is a general $\mathcal{EL}^{sa}$-KB, a an individual name in $\mathcal{A}$, and Σ a finite signature, then $\exists^{sim}\Sigma.(\mathcal{T}, \mathcal{A}, a)$ is an $\mathcal{EL}^{sa}$-concept;
- if C, D are $\mathcal{EL}^{sa}$-concepts, then $C \sqsubseteq D$ is an $\mathcal{EL}^{sa}$-concept inclusion;
- a general $\mathcal{EL}^{sa}$-TBox is a finite set of $\mathcal{EL}^{sa}$-concept inclusions;
- an $\mathcal{EL}^{sa}$-KB is a pair $(\mathcal{T}, \mathcal{A})$ consisting of a general $\mathcal{EL}^{sa}$-TBox and an ABox.

The semantics of $\exists^{sim}\Sigma.(\mathcal{T}, \mathcal{A}, a)$ is as follows:

> $d \in (\exists^{sim}\Sigma.(\mathcal{T}, \mathcal{A}, a))^{\mathcal{I}}$ iff there exists a model $\mathcal{J}$ of $(\mathcal{T}, \mathcal{A})$ such that $(\mathcal{J}, a^{\mathcal{J}}) \leq_{\Gamma} (\mathcal{I}, d)$, where $\Gamma = (\mathsf{N_C} \cup \mathsf{N_R}) \setminus \Sigma$.

Example 3. Let $\mathcal{T} = \emptyset$, $\mathcal{A} = \{r(a, a)\}$, and $\Sigma = \emptyset$. Then $\exists^{sim}\Sigma.(\mathcal{T}, \mathcal{A}, a)$ is equivalent to the concept $\exists^{sim}(\mathcal{I}, d)$ defined in Example 1.

Let $\mathcal{L}_1, \mathcal{L}_2$ be sets of concepts. We say that $\mathcal{L}_2$ is *polynomially at least as expressive as* $\mathcal{L}_1$, in symbols $\mathcal{L}_1 \leq_p \mathcal{L}_2$, if for every $C_1 \in \mathcal{L}_1$ one can construct in polynomial time a $C_2 \in \mathcal{L}_2$ such that C_1 and C_2 are equivalent. We say that $\mathcal{L}_1, \mathcal{L}_2$ are *polynomially equivalent*, in symbols $\mathcal{L}_1 \equiv_p \mathcal{L}_2$, if $\mathcal{L}_1 \leq_p \mathcal{L}_2$ and $\mathcal{L}_2 \leq_p \mathcal{L}_1$.

Theorem 3. *The languages $\mathcal{EL}^{\nu+}$, $\mathcal{EL}^{si}$, $\mathcal{EL}^{st}$, and $\mathcal{EL}^{sa}$ are polynomially equivalent.*

We provide sketches of proofs of $\mathcal{EL}^{si} \leq_p \mathcal{EL}^{\nu+}$, $\mathcal{EL}^{\nu+} \leq_p \mathcal{EL}^{st}$, $\mathcal{EL}^{st} \leq_p \mathcal{EL}^{sa}$, and $\mathcal{EL}^{sa} \leq_p \mathcal{EL}^{si}$.

$\mathcal{EL}^{si} \leq_p \mathcal{EL}^{\nu+}$. By Lemma 1, considering $\mathcal{EL}^{si}$-concepts of the form $\exists^{sim}(\mathcal{I}, d)$ is sufficient. Each such concept is equivalent to the $\mathcal{EL}^{\nu+}$-concept $\nu_\ell d_1 \cdots d_n.C_1, \ldots, C_n$, where $\Delta^{\mathcal{I}} = \{d_1, \ldots, d_n\}$ is regarded as a set of concept variables, $d = d_\ell$, and

$$C_i = \bigsqcap\{A \mid d_i \in A^{\mathcal{I}}\} \sqcap \bigsqcap\{\exists r.d_j \mid (d_i, d_j) \in r^{\mathcal{I}}\}.$$

$\mathcal{EL}^{\nu+} \leq_p \mathcal{EL}^{st}$. Let C be a closed $\mathcal{EL}^{\nu+}$-concept. An equivalent $\mathcal{EL}^{st}$-concept is constructed by replacing each subconcept of C of the form $\nu_\ell X_1, \ldots, X_n.C_1, \ldots, C_n$ with an $\mathcal{EL}^{st}$-concept, proceeding from the inside out. We assume that for every variable X that occurs in the original $\mathcal{EL}^{\nu+}$-concept C, there is a concept name A_X that does not occur in C. Now $\nu_\ell X_1, \ldots, X_n.C_1, \ldots, C_n$ (which potentially contains free variables) is replaced with the $\mathcal{EL}^{st}$-concept

$$\exists^{sim}\{A_{X_1}, \ldots, A_{X_n}\}.(\{A_{X_i} \sqsubseteq C_i^{\downarrow} \mid 1 \leq i \leq n\}, A_{X_\ell})$$

where $C_i^{\downarrow}$ is obtained from C_i by replacing every variable X with the concept name A_X.

$\mathcal{EL}^{st} \leq_p \mathcal{EL}^{sa}$. Let C be an $\mathcal{EL}^{st}$-concept. As already observed, we may assume that D is a concept name in all subconcepts $\exists^{sim}\Sigma.(\mathcal{T}, D)$ of C. Now replace each

$\exists^{sim}\Sigma.(\mathcal{T}, A)$ in C, proceeding from the inside out, by $\exists^{sim}\Sigma.(\mathcal{T}, \mathcal{A}, a)$, where $\mathcal{A} = \{A(a)\}$. The resulting concept is equivalent to C.

$\mathcal{EL}^{sa} \leq_p \mathcal{EL}^{si}$. To prove this inclusion, we make use of *canonical models* for $\mathcal{EL}^{sa}$-KBs, similar to those used for $\mathcal{EL}$ in [4]. In particular, canonical models for $\mathcal{EL}^{sa}$ can be constructed by an extension of the algorithm given in [4], see the full version for details.

Theorem 4 (Canonical model). *For every satisfiable $\mathcal{EL}^{sa}$-KB $(\mathcal{T}, \mathcal{A})$, one can construct in polynomial time a model $\mathcal{I}_{\mathcal{T},\mathcal{A}}$ of $(\mathcal{T}, \mathcal{A})$ with $|\Delta^{\mathcal{I}_{\mathcal{T},\mathcal{A}}}|$ bounded by twice the size of $(\mathcal{T}, \mathcal{A})$ and such that for every model $\mathcal{J}$ of $(\mathcal{T}, \mathcal{A})$, we have $(\mathcal{I}_{\mathcal{T},\mathcal{A}}, a^{\mathcal{I}_{\mathcal{T},\mathcal{A}}}) \leq (\mathcal{J}, a^{\mathcal{J}})$ for all $a \in \mathsf{Ind}(\mathcal{A})$.*

To prove $\mathcal{EL}^{sa} \leq_p \mathcal{EL}^{si}$, it suffices to show that any outermost occurrence of a concept of the form $\exists^{sim}\Sigma.(\mathcal{T}, \mathcal{A}, a)$ in an $\mathcal{EL}^{sa}$-concept C can be replaced with the equivalent $\mathcal{EL}^{si}$-concept $\exists^{sim}(\mathcal{I}^{\Sigma}_{\mathcal{T},\mathcal{A}}, a)$, where $\mathcal{I}^{\Sigma}_{\mathcal{T},\mathcal{A}}$ denotes $\mathcal{I}_{\mathcal{T},\mathcal{A}}$ except that all $\sigma \in \Sigma$ are interpreted as empty sets. First let $d \in (\exists^{sim}\Sigma.(\mathcal{T}, \mathcal{A}, a))^{\mathcal{J}}$. Then there is a model $\mathcal{I}'$ of $(\mathcal{T}, \mathcal{A})$ such that $(\mathcal{I}', a^{\mathcal{I}'}) \leq_\Sigma (\mathcal{J}, d)$. By Theorem 4, $(\mathcal{I}_{\mathcal{T},\mathcal{A}}, a^{\mathcal{I}_{\mathcal{T},\mathcal{A}}}) \leq (\mathcal{I}', a^{\mathcal{I}'})$. Thus, by closure of simulations under composition, $(\mathcal{I}^{\Sigma}_{\mathcal{T},\mathcal{A}}, a) \leq_\Sigma (\mathcal{J}, d)$ as required. The converse direction follows from the condition that $\mathcal{I}_{\mathcal{T},\mathcal{A}}$ is a model of $(\mathcal{T}, \mathcal{A})$. This finishes our proof sketch for Theorem 3.

It is interesting to note that, as a consequence of the proofs of Theorem 3, for every $\mathcal{EL}^{\nu+}$-concept there is an equivalent $\mathcal{EL}^{\nu+}$-concept of polynomial size in which the greatest fixpoint constructor is not nested, and similarly for $\mathcal{EL}^{st}$, $\mathcal{EL}^{sa}$. An important consequence of the existence of canonical models, as granted by Theorem 4, is that reasoning in our family of extensions of $\mathcal{EL}$ is tractable.

Theorem 5 (Tractable reasoning). *Let $\mathcal{L}$ be any of the languages $\mathcal{EL}^{\nu}$, $\mathcal{EL}^{\nu+}$, $\mathcal{EL}^{si}$, $\mathcal{EL}^{st}$, or $\mathcal{EL}^{sa}$. Then KB consistency, subsumption w.r.t. TBoxes, and the instance problem can be decided in* PTIME.

Proof. By Theorem 3, it suffices to concentrate on $\mathcal{L} = \mathcal{EL}^{sa}$. Consistency can be decided in PTIME by the algorithm that constructs the canonical model. Subsumption can be polynomially reduced in the standard way to the instance problem. Finally, by Theorem 4, we can decide the instance problem as follows: to decide whether $(\mathcal{T}, \mathcal{A}) \models C(a)$, where we can w.l.o.g. assume that $C = A$ for a concept name A, we check whether $(\mathcal{T}, \mathcal{A})$ is inconsistent or $a^{\mathcal{I}_{\mathcal{T},\mathcal{A}}} \in A^{\mathcal{I}_{\mathcal{T},\mathcal{A}}}$. Both can be done in PTIME. □

5 Characterizing $\mathcal{EL}^{\nu}$ using simulations

When characterizing $\mathcal{EL}$ as a fragment of first-order logic in Theorem 2, our starting point was the observation that $\mathcal{EL}$-concepts are preserved under simulations and that $\mathcal{EL}$ is a Horn logic, thus having finite minimal models. The same is true for $\mathcal{EL}^{\nu}$: first, $\mathcal{EL}^{\nu}$-concepts are preserved under simulations, as $\mathcal{EL}^{si}$ is obviously preserved under simulations and, by Theorem 3, every $\mathcal{EL}^{\nu}$-concept is equivalent to an $\mathcal{EL}^{si}$-concept. And second, a finite minimal model of an $\mathcal{EL}^{\nu}$-concept C can be constructed by taking the canonical model $\mathcal{I}_{\mathcal{T},\mathcal{A}}$ from Theorem 4 for $\mathcal{T} = \{A \sqsubseteq C\}$ and $\mathcal{A} = \{A(a)\}$. As required, we then have $\models C \sqsubseteq D$ iff $(\mathcal{T}, \mathcal{A}) \models D(a)$ iff $a \in D^{\mathcal{I}_{\mathcal{T},\mathcal{A}}}$, for all $\mathcal{EL}^\nu$-concepts D. However, $\mathcal{EL}^\nu$ is clearly not a fragment of FO. Instead, it relates to *MSO* in exactly the way that $\mathcal{EL}$ related to FO.

Theorem 6. *The set of $\mathcal{EL}^\nu$-concepts is a maximal set of MSO-formulas that is preserved under simulations and has finite minimal models: if $\mathcal{L}$ is a set of MSO-formulas that properly contains all $\mathcal{EL}^\nu$-concepts, then either it contains a formula not preserved under simulations or is does not have finite minimal models.*

Proof. Assume that $\mathcal{L} \supseteq \mathcal{EL}^\nu$ is preserved under simulations and has finite minimal models. Let $\varphi(x) \in \mathcal{L}$. We have to show that $\varphi(x)$ is equivalent to an $\mathcal{EL}^\nu$-concept. To this end, take a finite minimal model of φ, i.e., an interpretation $\mathcal{I}$ and a $d \in \Delta^\mathcal{I}$ such that for all $\psi(x) \in \mathcal{L}$ we have that $\forall.(\varphi(x) \rightarrow \psi(x))$ is valid iff $\mathcal{I} \models \psi[d]$. We will show that φ is equivalent to (the MSO translation of) $\exists^{sim}(\mathcal{I}, d)$. We may assume that $\exists^{sim}(\mathcal{I}, d) \in \mathcal{L}$. Since $d \in (\exists^{sim}(\mathcal{I}, d))^\mathcal{I}$, we thus have that $\forall x.(\varphi(x) \rightarrow \exists^{sim}(\mathcal{I}, d)(x))$ is valid. Conversely, assume that $d' \in (\exists^{sim}(\mathcal{I}, d))^\mathcal{J}$ for some interpretation $\mathcal{J}$. Then $(\mathcal{I}, d) \leq (\mathcal{J}, d')$. We have $(\mathcal{I}, d) \models \varphi[d]$. Thus, by preservation of $\varphi(x)$ under simulations, $\mathcal{J} \models \varphi[d']$. Thus $\forall x.(\exists^{sim}(\mathcal{I}, d)(x) \rightarrow \varphi(x))$ is also valid. □

A number of closely related characterizations remain open. For example, we conjecture that an extension of Theorem 1 holds for $\mathcal{EL}^{\nu,\sqcup}$ and MSO (instead of $\mathcal{EL}$ and FO). Also, it is open whether Theorem 6 still holds if finite minimal models are replaced by arbitrary minminal models.

6 Applications and Logical Properties

The μ-calculus is considered to be extremely well-behaved regarding its expressive power and logical properties. The aim of this section is to take a brief look at the expressive power of its $\mathcal{EL}$-analogues $\mathcal{EL}^\nu$ and $\mathcal{EL}^{\nu+}$. In particular, we show that $\mathcal{EL}^{\nu+}$ is more well-behaved than $\mathcal{EL}$ in a number of respects. Throughout this section, we will not distinguish between the languages previously proved polynomially equivalent.

A first observation concerns the most specific concept, which plays a role in the bottom-up construction of knowledge bases and has received quite a bit of attention in the context of $\mathcal{EL}$ [2, 6]. Formally, a concept C is the *most specific concept (MSC)* for an individual a in a knowledge base $(\mathcal{T}, \mathcal{A})$ if

- $(\mathcal{T}, \mathcal{A}) \models C(a)$ and
- for every concept D with $(\mathcal{T}, \mathcal{A}) \models D(a)$, we have $\mathcal{T} \models C \sqsubseteq D$.

In $\mathcal{EL}$, the MSC need not exist, as is witnessed by the knowledge base $(\emptyset, \{r(a, a)\})$, where the MSC for a is non-existent. In $\mathcal{EL}^\nu$, the MSC always exists for any a in any $(\mathcal{T}, \mathcal{A})$, as it is simply $\exists^{sim}\emptyset.(\mathcal{T}, \mathcal{A}, a)$. Argueably, this is more straightforward than existing definitions of the MSC in extensions of $\mathcal{EL}$ with *fixpoint TBoxes*, which need to refer to conservative extensions [2]—see Section 7 for more details.

We now turn our attention to issues of definability and interpolation. From now on, we use $\mathsf{sig}(C)$ to denote the set of concept and role names used in the concept C.

A concept C is a *Σ-concept* if $\mathsf{sig}(C) \subseteq \Sigma$. Let $\mathcal{T}$ be a general $\mathcal{EL}^{\nu+}$-TBox, C an $\mathcal{EL}^{\nu+}$-concept and Γ a finite signature.

We start with considering the fundamental notion of a Γ-definition. The question addressed here is whether a given concept can be expressed in an equivalent way by referring only to the symbols in a given signature Γ. It has been convincingly argued in [20, 19] that signature manipulations of this sort are fundamental for structuring TBoxes and for ontology based data access. Formally, a Γ-concept D is an *explicit Γ-definition* of a concept C w.r.t. a TBox $\mathcal{T}$ if, and only if, $\mathcal{T} \models C \equiv D$ (i.e., C and D are equivalent w.r.t. $\mathcal{T}$). Clearly, explicit Γ-definitions do not always exist in any of the logics studied in this paper: for example, there is no explicit $\{A\}$-definition of B w.r.t. the TBox $\{A \sqsubseteq B\}$. However, it is not hard to show the following using the fact that $\exists^{sim}\Sigma.(\mathcal{T}, C)$ is the most specific Γ-concept that subsumes C w.r.t. $\mathcal{T}$.

Proposition 2. *Let C be an $\mathcal{EL}^{\nu+}$-concept, $\mathcal{T}$ a general $\mathcal{EL}^{\nu+}$-TBox and Γ a signature. There exists an explicit Γ-definition of C w.r.t. $\mathcal{T}$ iff $\exists^{sim}\Sigma.(\mathcal{T}, C)$ is such a definition (for $\Sigma = \mathsf{sig}(\mathcal{T}, C) \setminus \Gamma$).*

It is interesting to note that if $\mathcal{T}$ happens to be a general $\mathcal{EL}$-TBox and C an $\mathcal{EL}$-concept and there exists an explicit Γ-definition of C w.r.t. $\mathcal{T}$, then the concept $\exists^{sim}\Sigma.(\mathcal{T}, C)$ from Proposition 2 is equivalent w.r.t. $\mathcal{T}$ to an $\mathcal{EL}$-concept over Γ. This follows from the fact that $\mathcal{EL}$ has the Beth definability property (see below for a definition) which follows immediately from interpolation results proved for $\mathcal{EL}$ in [14]. The advantage of giving explicit Γ-definitions in $\mathcal{EL}^{\nu+}$ even when $\mathcal{T}$ and C are formulated in $\mathcal{EL}$ is that Γ-definitions in $\mathcal{EL}^{\nu+}$ are of polynomial size while the following example shows that they may be exponentially large in $\mathcal{EL}$.

Example 4. Let $\mathcal{T}$ consist of $A_i \equiv \exists r_i.A_{i+1} \sqcap \exists s_i A_{i+1}$ for $0 \leq i \leq n$. $A_n \equiv \top$. Let $\Gamma = \{r_0, \ldots, r_{n-1}, s_0, \ldots, s_{n-1}\}$. Then A_0 has an explicit Γ-definition w.r.t. $\mathcal{T}$ in $\mathcal{EL}$, namely C_0, where $C_i = \exists r_i.C_{i+1} \sqcap \exists s_i.C_{i+1}$ and $C_n = \top$. This definition is of exponential size and it is easy to see that there is no shorter Γ-definition of A_0 w.r.t. $\mathcal{T}$ in $\mathcal{EL}$.

Say that a concept C is *implicitly Γ-defined w.r.t. $\mathcal{T}$* iff $\mathcal{T} \cup \mathcal{T}_\Gamma \models C \equiv C_\Gamma$, where $\mathcal{T}_\Gamma$ and C_Γ are obtained from $\mathcal{T}$ and C, respectively, by replacing each $\sigma \notin \Gamma$ by a fresh symbol σ'. The Beth definability property, which was studied in a DL context in [20, 19], ensures that explicit Γ-definitions always exist when they possibly can.

Theorem 7 (Beth Property). *$\mathcal{EL}^{\nu+}$ has the polynomial Beth definability property: for every general $\mathcal{EL}^{\nu+}$-TBox $\mathcal{T}$, concept C, and signature Γ such that C is implicitly Γ-defined w.r.t. $\mathcal{T}$, there is an explicit Γ-definition w.r.t. $\mathcal{T}$, namely $\exists^{sim}(\mathsf{sig}(\mathcal{T}, C) \setminus \Gamma).(\mathcal{T}, C)$.*

The proof of Theorem 7 relies on $\mathcal{EL}^{\nu}$ having a certain interpolation property. Say that two general TBoxes $\mathcal{T}_1$ and $\mathcal{T}_2$ are *Δ-inseparable* w.r.t. $\mathcal{EL}^{\nu}$ if $\mathcal{T}_1 \models C \sqsubseteq D$ iff $\mathcal{T}_2 \models C \sqsubseteq D$ for all $\mathcal{EL}^{\nu}$-inclusions $C \sqsubseteq D$.

Theorem 8 (Interpolation). *Let $\mathcal{T}_1 \cup \mathcal{T}_2 \models C \sqsubseteq D$ and assume that $\mathcal{T}_1$ and $\mathcal{T}_2$ are Δ-inseparable w.r.t. $\mathcal{EL}^{\nu}$ for $\Delta = \mathsf{sig}(\mathcal{T}_1, C) \cap \mathsf{sig}(\mathcal{T}_2, D)$. Then the Δ-concept $F = \exists^{sim}\Sigma.(\mathcal{T}_1, C)$, $\Sigma = \mathsf{sig}(\mathcal{T}_1, C) \setminus \Delta$, is an interpolant of C, D w.r.t. $\mathcal{T}_1, \mathcal{T}_2$; i.e. $\mathcal{T}_1 \models C \sqsubseteq F$ and $\mathcal{T}_2 \models F \sqsubseteq D$.*

We show how Theorem 7 follows from Theorem 8. Assume that $\mathcal{T} \cup \mathcal{T}_\Gamma \models C \equiv C_\Gamma$, where $\mathcal{T}, \mathcal{T}_\Gamma, C, C_\Gamma$ satisfy the conditions of Theorem 7. Then $\mathcal{T}$ and $\mathcal{T}_\Gamma$ are Γ-inseparable and $\Gamma \supseteq \mathsf{sig}(\mathcal{T}, C) \cap \mathsf{sig}(\mathcal{T}_\Gamma, C_\Gamma)$. Thus, by Theorem 8, $\mathcal{T} \models \exists^{sim}\Sigma.(\mathcal{T}_\Gamma, C_\Gamma) \sqsubseteq C$ for $\Sigma = \mathsf{sig}(\mathcal{T}_\Gamma, C_\Gamma) \setminus \Gamma$. Now Theorem 7 follows from the fact that $\exists^{sim}\Sigma.(\mathcal{T}_\Gamma, C_\Gamma)$ is equivalent to $\exists^{sim}\Sigma'.(\mathcal{T}, C)$ for $\Sigma' = \mathsf{sig}(\mathcal{T}, C) \setminus \Gamma$.

In [14], it is shown that $\mathcal{EL}$ also has this interpolation property. However, the advantage of using $\mathcal{EL}^{\nu+}$ is that interpolants are of polynomial size. The decomposition algorithm for $\mathcal{EL}$ given in [14] crucially depends on this property of $\mathcal{EL}^{\nu+}$.

7 Relation to TBoxes with Fixpoint Semantics

There is a tradition of considering DLs that introduce fixpoints at the TBox level instead of at the concept level [17, 18, 1]. In [3], Baader proposes and analyzes such a DL based on $\mathcal{EL}$ and greatest fixpoints. This DL, which we call $\mathcal{EL}^{\mathsf{gfp}}$ from now on, differs from our $\mathcal{EL}^{\nu}$ in that (i) TBoxes are classical TBoxes rather than sets of GCIs (but cycles are allowed) and (ii) the ν-concept constructor is not present; instead, a greatest fixpoint semantics is adopted for the defined concept names.

On the concept level, $\mathcal{EL}^{\nu}$ is clearly strictly more expressive than $\mathcal{EL}^{\mathsf{gfp}}$: since fixpoints are introduced at the TBox level, concepts of $\mathcal{EL}^{\mathsf{gfp}}$ coincide with $\mathcal{EL}$-concepts, and thus there is no $\mathcal{EL}^{\mathsf{gfp}}$-concept equivalent to the $\mathcal{EL}^{\nu}$-concept $\nu X.\exists r.X$. In the following, we show that $\mathcal{EL}^{\nu}$ is also more expressive than $\mathcal{EL}^{\mathsf{gfp}}$ also on the TBox level, even if we restrict $\mathcal{EL}^{\nu}$-TBoxes as in $\mathcal{EL}^{\mathsf{gfp}}$. We use the standard notion of logical equivalence, i.e., two TBoxes $\mathcal{T}$ and $\mathcal{T}'$ are *equivalent* iff $\mathcal{T}$ and $\mathcal{T}'$ have precisely the same models. As observed by Schild in the context of $\mathcal{ALC}$ [18], every $\mathcal{EL}^{\mathsf{gfp}}$-TBox $\mathcal{T} = \{A_1 \equiv C_1, \ldots, A_n \equiv C_n\}$ is equivalent in this sense to the $\mathcal{EL}^{\nu+}$-TBox $\{A_i \equiv \nu_i X_1, \ldots, X_n.C'_1, \ldots, C'_n \mid 1 \leq i \leq n\}$, where each C'_i is obtained from C_i by replacing each A_j with X_j, $1 \leq j \leq n$. Note that since we are translating to mutual fixpoints, the size of the resulting TBox is polynomial in the size of the original one. In the converse direction, there is no equivalence-preserving translation.

Lemma 2. *For each $\mathcal{EL}^{\mathsf{gfp}}$-TBox, there is an equivalent $\mathcal{EL}^{\nu+}$-TBox of polynomial size, but no $\mathcal{EL}^{\mathsf{gfp}}$-TBox is equivalent to the $\mathcal{EL}^{\nu}$-TBox $\{A \equiv P \sqcap \nu X.\exists r.X\}$.*

Proof. (sketch) It is not hard to prove that for every $\mathcal{EL}^{\mathsf{gfp}}$-TBox $\mathcal{T}$, defined concept name A in $\mathcal{T}$, and role name r, one of the following holds:

- there is an $m \geq 0$ such that $A \sqsubseteq \exists r^n.\top$ implies $n \leq m$ or
- $A \sqsubseteq \exists r^n.B$ for some $n > 0$ and defined concept name B.

However, no such TBox can be equivalent to $A \sqsubseteq \exists r^n.B$ since $\mathcal{T} \models \exists r^n.\top$ for all $n > 0$, but there is no $n > 0$ and defined concept name B with $A \sqsubseteq \exists r^n.B$. □

$\mathcal{EL}^{\mathsf{gfp}}$ and $\mathcal{EL}^{\nu}$ become equi-expressive if the strict notion of equivalence used above is replaced with one based on conservative extensions, thus allowing the introduction of new concept names that are suppressed from logical equivalence. However, we believe that not having to deal with conservative extensions is an advantage of $\mathcal{EL}^{\nu}$ over $\mathcal{EL}^{\mathsf{gfp}}$, as conservative extensions tend to make simple definitions somewhat awkward, c.f. the least common subsumers and most specific concepts for $\mathcal{EL}^{\mathsf{gfp}}$ in [2, 3].

References

1. F. Baader. Using automata theory for characterizing the semantics of terminological cycles. *Annals of Mathematics and Artificial Intelligence*, 18(2–4):175–219, 1996.
2. F. Baader. The instance problem and the most specific concept in the description logic $\mathcal{EL}$ w.r.t. terminological cycles with descriptive semantics. In *Proc. of KI 2003*, volume 2821 of *LNAI*, pages 64–78, 2003. Springer.
3. F. Baader. Terminological cycles in a description logic with existential restrictions. In *Proc. of IJCAI03*, pages 325–330. Morgan Kaufmann, 2003.
4. F. Baader, S. Brandt, and C. Lutz. Pushing the $\mathcal{EL}$ envelope. In *Proc. of IJCAI05*. Morgan Kaufmann, 2005.
5. F. Baader, D. Calvanes, D. McGuiness, D. Nardi, and P. Patel-Schneider. *The Description Logic Handbook: Theory, implementation and applications*. Cambridge Univ. Press, 2003.
6. F. Baader and F. Distel. A finite basis for the set of $\mathcal{EL}$-implications holding in a finite model. In *Proc. of ICFCA8*, volume 4933 of *LNAI*, pages 46–61. Springer, 2008.
7. J. Bradfield and C. Stirling. Modal mu-calculus. In *Handbook of Modal Logic*. Elsevier, 2006.
8. D. Calvanese, G. De Giacomo, D. Lembo, M. Lenzerini, and R. Rosati. Tractable reasoning and efficient query answering in description logics: The *DL-Lite* family. *J. of Automated Reasoning*, 39(3):385–429, 2007.
9. J. W. de Bakker. *Mathematical Theory of Program Correctness*. Prentice-Hall, 1980.
10. T. French. *Bisimulation quantifiers for modal logics*. PhD thesis, University of Western Australia, 2006.
11. G. De Giacomo and M. Lenzerini. Boosting the Correspondence between Description Logics and Propositional Dynamic Logics. In *Proc. of AAAI94*, pages 205-212. AAAI Press, 1994.
12. V. Goranko and M. Otto. Model theory of modal logic. In *Handbook of Modal Logic*. Elsevier, 2007.
13. D. Janin and I. Walukiewicz. On the expressive completeness of the propositional mu-calculus with respect to monadic second order logic. In *Proc. of CONCUR96*, volume 1119 of *PNCS*, pages 263–277. Springer, 1996.
14. B. Konev, C. Lutz, D. Ponomaryov, and F. Wolter. Decomposing description logic ontologies. In *Proceedings of KR10*, 2010.
15. N. Kurtonina and M. de Rijke. Expressiveness of concept expressions in first-order description logics. *Artificial Intelligence*, 107, 1999.
16. C. Lutz and F. Wolter. Deciding inseparability and conservative extensions in the description logic $\mathcal{EL}$. *J. of Symbolic Computation*, 45(2):194–228, 2010.
17. B. Nebel. Terminological cycles: Semantics and computational properties. In *Principles of Semantic Networks*. Morgan Kaufmann, 1991.
18. K. Schild. Terminological cycles and the propositional μ-calculus. In *Proc. of KR'94*, pages 509–520. Morgan Kaufmann, 1994.
19. I. Seylan, E. Franconi, and J. de Bruijn. Effective query rewriting with ontologies over DBoxes. In *Proc. of IJCAI09*, pages 923–925, 2009.
20. B. ten Cate, W. Conradie, M. Marx, and Y. Venema. Definitorially complete description logics. In *Proc. of KR'06*, pages 79–89. AAAI Press, 2006.
21. J. van Benthem. Modal Correspondence Theory. Mathematical Institute, University of Amsterdam, 1976.
22. M. Vardi and P. Wolper. Automata theoretic techniques for modal logics of programs. In *Proc. of STOC84*, pages 446–456. ACM Press, 1984.
23. M. Y. Vardi. Why is modal logic so robustly decidable? In *Descriptive Complexity and Finite Models*, pages 149–184, 1996.

A Proofs for Section 3

In this section, we prove Theorems 1 and 2. We require some basic and well-known operations on interpretations. First, for an interpretation $\mathcal{I}$ and $d \in \Delta^{\mathcal{I}}$, we denote by $\mathcal{I}_d$ the *tree-unraveling* of $\mathcal{I}$ in d: the domain $\Delta^{\mathcal{I}_d}$ consists of all words

$$dr_1 d_1 r_2 \cdots r_n d_n$$

such that $n \geq 0$ and $(d_i, d_{i+1}) \in r_{i+1}^{\mathcal{I}}$ for all $i \geq 0$ (we set $d_0 = d$). We let $\sigma \cdot d' \in A^{\mathcal{I}_d}$ iff $d' \in A^{\mathcal{I}}$ and $r^{\mathcal{I}_d}$ consists of all pairs $(\sigma, \sigma r d') \in \Delta^{\mathcal{I}_d} \times \Delta^{\mathcal{I}_d}$.

Secondly, for $\ell \geq 0$, we denote with $\mathcal{I}_d^\ell$ the subinterpretation of $\mathcal{I}_d$ induced by all elements that are reachable in at most ℓ-many steps from the root d.

Definition 2 (ℓ-local). *A concept C is called ℓ-local iff there is some $\ell \geq 0$ such that for all pointed interpretations $(\mathcal{I}, d)$ we have $d \in C^{\mathcal{I}}$ iff $d \in C^{\mathcal{I}_d^\ell}$.*

Let Σ be a finite signature. A *pointed Σ-interpretation $\mathcal{I}$* is an interpretation in which $\sigma^{\mathcal{I}} = \emptyset$ for all $\sigma \notin \Sigma$.

Definition 3. *Let Σ be finite, and $(\mathcal{I}, d)$ a pointed Σ-interpretation. The ℓ-characteristic concept $X^\ell(\mathcal{I}, d)$ of $(\mathcal{I}, d)$ is recursively defined as follows:*

$$X^0(\mathcal{I}, d) := \bigsqcap\{A \in \mathsf{N_C} \mid d \in A^{\mathcal{I}}\}$$
$$X^{\ell+1}(\mathcal{I}, d) := X^0(\mathcal{I}, d) \sqcap \bigsqcap_{r \in \mathsf{N_R}} \bigsqcap\{\exists r.X^\ell(\mathcal{I}, d') \mid (d, d') \in r^{\mathcal{I}}\}$$

Observe that for $X^{\ell+1}(\mathcal{I}, d)$ to be well-defined, we have to ensure that there are only finitely many conjuncts in X when forming a conjunction $\bigsqcap_{C \in X} C$. This can be easily proved by induction. In fact, one can easily prove that, up to logical equivalence, there exist only finitely many ℓ-characteristic concepts for each finite signature.

Observation 1 *For each finite signature Σ and each $\ell \geq 0$ there are, up to logical equivalence, finitely many ℓ-characteristic concepts for Σ.*

Observation 2 *$e \in (X^\ell(\mathcal{I}, d))^{\mathcal{J}}$ iff $(\mathcal{I}_d^\ell, d) \leq (\mathcal{J}, e)$.*

Proof. We start with the direction from left to right. The proof is by induction on ℓ. For $\ell = 0$, the claim is trivial. Now assume that it has been proved for ℓ and let $e \in (X^{\ell+1}(\mathcal{I}, d))^{\mathcal{J}}$. We may assume that $\mathcal{I}$ is already a tree-structure. Then for all $r \in \mathsf{N_R}$ and every d' with $(d, d') \in r^{\mathcal{I}}$, we have $e \in (\exists r.X^\ell(\mathcal{I}, d'))^{\mathcal{J}}$. So there is some e' with $(e, e') \in r^{\mathcal{J}}$ such that $e' \in (X^\ell(\mathcal{I}, d'))^{\mathcal{J}}$. The induction hypothesis yields that there is a simulation $S_{d'} : (\mathcal{I}_{d'}^\ell, d') \leq (\mathcal{J}, e')$. We define

$$S_d := \{(d, e)\} \cup \bigcup_{r \in \Sigma} \bigcup_{(d,d') \in r^{\mathcal{I}}} S_{d'}$$

and show that S_d is a simulation.

Condition 1 of Definition 1 holds for all $(d', e') \neq (d, e)$ in S_d as they already belong to simulations. Additionally, $e \in (X^0(\mathcal{I}, d))^{\mathcal{J}}$ and so $d \in A^{\mathcal{I}}$ implies $e \in A^{\mathcal{J}}$ for all concept names $A \in \Sigma$, too.

Condition 2 is met for (d, e): By definition of S_d, for all $r \in \Sigma$ and every r-successor d' of d there is an r-successor e' of e such that $(d', e') \in S_d$.

Let $(d_1, e_1) \in S$ and let $(d, e) \neq (d_1, e_1)$. So $(d_1, e_1) \in S_{d'}$ for some r-successor d' of d. Now let $s \in \Sigma$ and d_2 be an s-successor of d_1 in $\mathcal{I}_d^{\ell+1}$. It is to show, that there is some s-successor e_2 of e_1 such that $(d_2, e_2) \in S_d$.

The tree interpretation $(\mathcal{I}_d^{\ell+1}, d)$ is composed of d, connected by edges to the roots of the subtrees $\mathcal{I}_{d'}^{\ell}$, obtained from its successors d'. So every s-successor of d_1 in $\mathcal{I}_d^{\ell+1}$ is an s-successor of d_1 in $\mathcal{I}_{d'}^{\ell}$. As $S_{d'}$ is a simulation, there must be some s-successor e_2 of e_1 such that $(d_2, e_2) \in S_{d'}$; hence $(d_2, e_2) \in S_d$.

For the converse direction, assume that $(\mathcal{I}_d^{\ell}, d) \leq (\mathcal{J}, e)$. Note that $d \in (X^{\ell}(\mathcal{I}, d))^{\mathcal{I}_d}$ and that $X^{\ell}(\mathcal{I}, d)$ has nesting-depth not exceeding ℓ and is therefore ℓ-local. Hence $d \in (X^{\ell}(\mathcal{I}, d))^{\mathcal{I}_d^{\ell}}$, which, as $\mathcal{EL}$-concepts are preserved by simulation, yields $e \in (X^{\ell}(\mathcal{I}, d))^{\mathcal{J}}$.

Proof (of Theorem 1). Let Σ denote the signature of a FO-formula $\varphi(x)$ what is preserved under simulation. It follows that $\varphi(x)$ is invariant under bisimulations, and so it can be regarded as an $\mathcal{ALC}$-concept F. We may assume that F contains symbols from Σ only. Clearly, every $\mathcal{ALC}$-concept is ℓ-local for ℓ the nesting-depth of existential and value restrictions in F. We define

$$X_F := \{X^{\ell}(\mathcal{I}_d^{\ell}, d) \mid d \in F^{\mathcal{I}}, \mathcal{I} \text{ a } \Sigma\text{-interpretation }\}.$$

According to Observation 1, on each level ℓ, there are up to logical equivalence only finitely many ℓ-characteristic concepts for Σ; thus, we may assume that X_F is finite and $\bigsqcup_{C \in X_F} C$ is well defined. It remains to be shown that F and $\bigsqcup_{C \in X_F} C$ are equivalent. To this end, it is sufficient to show that $e \in F^{\mathcal{J}}$ iff $e \in (\bigsqcup_{C \in X_F} C)^{\mathcal{J}}$ for all pointed Σ-interpretations $(\mathcal{J}, e)$. Let $e \in F^{\mathcal{J}}$. We may assume that $X^{\ell}(\mathcal{J}_e^{\ell}, e) \in X_F$. As $(\mathcal{J}_e^{\ell}, e) \leq (\mathcal{J}, e)$, Observation 2 yields $e \in (X^{\ell}(\mathcal{J}_e^{\ell}, e))^{\mathcal{J}}$, which entails $e \in (\bigsqcup_{C \in X_F} C)^{\mathcal{J}}$.

Conversely, let $e \in (\bigsqcup_{C \in X_F} C)^{\mathcal{J}}$. Then $e \in (X^{\ell}(\mathcal{I}_d^{\ell}, d))^{\mathcal{J}}$ for some $X^{\ell}(\mathcal{I}_d^{\ell}, d) \in X_F$. From Observation 2, we obtain $(\mathcal{I}_d^{\ell}, d) \leq (\mathcal{J}, e)$. Due to the ℓ-locality of F, we have $d \in F^{\mathcal{I}_d^{\ell}}$ and as F is simulation preserved, $e \in F^{\mathcal{J}}$.

Proof (of Theorem 2). Assume that $\mathcal{L}$ is a fragment of FO containing $\mathcal{EL}$ that is preserved under simulations and has (finite) minimal models. Let $\varphi(x) \in \mathcal{L}$. We show $\varphi(x)$ is equivalent to an $\mathcal{EL}$-concept. $\varphi(x)$ is logically equivalent to $\bigsqcup_{C \in X_F} C$, where X_F is as introduced in the proof of Theorem 1. Note that X_F contains $\mathcal{EL}$-concepts only. Thus, it is sufficient to show that $\bigsqcup_{C \in X_F} C$ is logically equivalent to one of its disjuncts.

Assume otherwise. Then $\forall x(\varphi(x) \rightarrow C(x))$ is not a tautology for any $C \in X_F$, where $C(x)$ denotes the translation of C into FO. Then $d \notin C^{\mathcal{I}}$ for any $C \in X_F$, where $(\mathcal{I}, d)$ is the (finite) minimal model of $\varphi(x)$. Thus $d \notin (\bigsqcup_{C \in X_F} C)^{\mathcal{I}}$ and we have derived a contradiction.

B Proofs for Section 4

To prove the claims for the translations to and from $\mathcal{EL}^{\nu+}$ we remind the reader of the following well-known characterization of the greatest fixpoint operator. For a fixpoint concept $\nu X.C$, we write $C(\nu X.C)$ to denote the result of unfolding the fixpoint once, i.e., replacing every occurrence of X in C with $(\nu X.C)$. Similarly, the expression $C_i(\nu_i X_1 \cdots X_n.C_1,\ldots,C_n)$ denotes the result of replacing every occurrence of X_j in C_i, $1 \leq j \leq n$, with $\nu_j X_1 \cdots X_n.C_1,\ldots,C_n$. We will consider concepts $\nu^\alpha X.C$ and $\nu_i^\alpha X_1 \cdots X_n.C_1,\ldots,C_n$ for every ordinal α. The semantics is defined by transfinite induction as

$$
\begin{aligned}
(\nu_i^0 X_1 \cdots X_n.C_1,\ldots,C_n)^{\mathcal{I},\mathcal{V}} &= \Delta^{\mathcal{I}}\\
(\nu_i^{\alpha+1} X_1 \cdots X_n.C_1,\ldots,C_n)^{\mathcal{I},\mathcal{V}} &= (C_i(\nu_i^\alpha X_1 \cdots X_n.C_1,\ldots,C_n))^{\mathcal{I},\mathcal{V}}\\
(\nu_i^\lambda X_1 \cdots X_n.C_1,\ldots,C_n)^{\mathcal{I},\mathcal{V}} &= \textstyle\bigcap_{\alpha<\lambda}(\nu_i^\alpha X_1 \cdots X_n.C_1,\ldots,C_n)^{\mathcal{I},\mathcal{V}}
\end{aligned}
$$

where λ is a limit ordinal. It is standard to show that for all interpretations, $d \in \Delta^{\mathcal{I}}$, and assignments $\mathcal{V}$, we have that $d \in (\nu_i X_1 \cdots X_n.C_1,\ldots,C_n)^{\mathcal{I},\mathcal{V}}$ if, and only if, $d \in (\nu_i^\alpha X_1 \cdots X_n.C_1,\ldots,C_n)^{\mathcal{I},\mathcal{V}}$ for all $\alpha < |\Delta^{\mathcal{I}}|$.

Lemma 3. $\mathcal{EL}^{si} \leq_p \mathcal{EL}^{\nu+}$.

Proof. We show that the claim that the concept $\exists^{sim}(\mathcal{I},d)$ is equivalent to the $\mathcal{EL}^{\nu+}$-concept $\nu_\ell d_1 \cdots d_n.C_1,\ldots,C_n$, where $\Delta^{\mathcal{I}} = \{d_1,\ldots,d_n\}$ is regarded as a set of concept variables, $d = d_\ell$, and

$$
C_i = \bigsqcap_{A\in\mathsf{N_C}}\{A \mid d_i \in A^{\mathcal{I}}\} \sqcap \bigsqcap_{r\in\mathsf{N_R}}\{\exists r.d_j \mid (d_i,d_j) \in r^{\mathcal{I}}\},
$$

is correct.

Let $\mathcal{J}$ be an interpretation and $e \in (\exists^{sim}(\mathcal{I},d))^{\mathcal{J}}$. Then $(\mathcal{I},d) \leq (\mathcal{J},e)$. We show by transfinite induction on α that for all $f \in \Delta^{\mathcal{J}}$ and $d_i \in \Delta^{\mathcal{I}}$ with $(\mathcal{I},d_i) \leq (\mathcal{J},f)$, we have $f \in (\nu_i^\alpha d_1 \cdots d_n.C_1,\ldots,C_n)^{\mathcal{J}}$. Clearly, this yields $e \in (\nu_\ell d_1 \cdots d_n.C_1,\ldots,C_n)^{\mathcal{J}}$ as required. Since the induction start is trivial, we concentrate on the induction step and the transfinite step. Let $f \in \Delta^{\mathcal{J}}$ and $d_i \in \Delta^{\mathcal{I}}$ with $(\mathcal{I},d_i) \leq (\mathcal{J},f)$.

- $f \in (\nu_i^{\alpha+1}\ell d_1 \cdots d_n.C_1,\ldots,C_n)^{\mathcal{J}}$.
  Let $(r_1,d_{i_1}),\ldots,(r_m,d_{i_m})$ be those elements of $\mathsf{N_R} \times \Delta^{\mathcal{I}}$ such that $(d_i,d_{i_j}) \in r_j^{\mathcal{I}}$ for $1 \leq j \leq m$. Since $(\mathcal{I},d_i) \leq (\mathcal{J},f)$, we find $f_1,\ldots,f_m \in \Delta^{\mathcal{J}}$ such that $(f,f_j) \in r_j^{\mathcal{J}}$ and $(\mathcal{I},d_{i_j}) \leq (\mathcal{J},f_j)$ for $1 \leq j \leq m$. The induction hypothesis yields $f_j \in (\nu_{i_j}^\alpha d_1 \cdots d_n.C_1,\ldots,C_n)^{\mathcal{J}}$. By definition of the $C_1,\ldots,C_n$ and the semantics of $\nu_i^{\alpha+1} d_1 \cdots d_n.C_1,\ldots,C_n$, this yields $f \in (\nu_i^{\alpha+1} d_1 \cdots d_n.C_1,\ldots,C_n)^{\mathcal{J}}$ as required.
- $f \in (\nu_i^\lambda d_1 \cdots d_n.C_1,\ldots,C_n)^{\mathcal{J}}$, λ a limit ordinal.
  By IH, we have that $f \in (\nu_i^\alpha d_1 \cdots d_n.C_1,\ldots,C_n)^{\mathcal{J}}$ for all $\alpha < \lambda$, thus it remains to use the semantics of $\nu_i^\lambda d_1 \cdots d_n.C_1,\ldots,C_n$.

Conversely, let $\mathcal{J}$ be an interpretation and $e \in (\nu_\ell d_1 \cdots d_n.C_1, \ldots, C_n)^{\mathcal{J}}$. We construct a sequence $S_0 \subseteq S_1 \subseteq \cdots$ of relations on $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{J}}$ such that

$$(d_k, f) \in S_i \text{ implies } f \in (\nu_k d_1 \cdots d_n.C_1, \ldots, C_n)^{\mathcal{J}}. \tag{$*$}$$

Start with putting $S_0 = \{(d_\ell, e)\}$. For the induction step, first set $S_{i+1} = S_i$. Then further extend S_{i+1} by considering all $(d_k, f) \in S_i \setminus S_{i-1}$ (where $S_{-1} := \emptyset$). Let $(r_1, d_{i_1}), \ldots, (r_m, d_{i_m})$ be those elements of $\mathsf{N_R} \times \Delta^{\mathcal{I}}$ such that $(d_k, d_{i_j}) \in r_j^{\mathcal{I}}$ for $1 \leq j \leq m$. We have $f \in (\nu_k d_1 \cdots d_n.C_1, \ldots, C_n)^{\mathcal{J}}$, thus also $f \in (C_k(\nu_k d_1 \cdots d_n.C_1, \ldots, C_n))^{\mathcal{J}}$. By definition of C_k, we thus find $f_1, \ldots, f_m \in \Delta^{\mathcal{J}}$ such that $(f, f_j) \in r_j$ and $f_j \in (\nu_{i_j} d_1 \cdots d_n.C_1, \ldots, C_n)^{\mathcal{J}}$ for $1 \leq j \leq m$. Add (f_j, d_{i_j}) to S_{i+1} for $1 \leq j \leq m$. Clearly, $(*)$ is satisfied.

Finally, set $S := \bigcup_{i \geq 0} S_i$. Using the definition of the S_i, $(*)$, and the definition of the concepts $C_1, \ldots, C_n$, it is straightforward to show that S is a simulation between $(\mathcal{I}, d_\ell)$ and $(\mathcal{J}, e)$. Thus, $e \in (\exists^{sim}(\mathcal{I}, d_\ell))^{\mathcal{J}}$ as required.

We call an $\mathcal{EL}^{\nu+}$-concept C with free variables $X_1, \ldots, X_k$ *equivalent* to an $\mathcal{EL}^{st}$-concept D if for all interpretations $\mathcal{I}$, $d \in \Delta^{\mathcal{I}}$, and assignments $\mathcal{V}$, we have $d \in C^{\mathcal{I},\mathcal{V}}$ iff $d \in D^{\mathcal{J}}$, where $\mathcal{J}$ is obtained from $\mathcal{I}$ by setting $A_X^{\mathcal{J}} = \mathcal{V}(X)$ for all variables X.

Lemma 4. $\mathcal{EL}^{\nu+} \leq_p \mathcal{EL}^{st}$.

Proof. The concept $C^\sharp$ is produced by starting with C and then replacing each subconcept of the form $\nu_\ell X_1, \ldots, X_n.C_1, \ldots, C_n$ with an $\mathcal{EL}^{st}$-concept, proceeding from the inside out. We assume that for every variable X that occurs in the original $\mathcal{EL}^{\nu+}$-concept C, there is a concept name A_X that does not occur in C.

We replace each subconcept $\nu_\ell X_1, \ldots, X_n.C_1, \ldots, C_n$ (which potentially contains free variables) with the $\mathcal{EL}^{st}$-concept

$$\exists^{sim}\{A_{X_1}, \ldots, A_{X_n}\}.(\{A_{X_i} \sqsubseteq C_i^\downarrow \mid 1 \leq i \leq n\}, A_{X_\ell})$$

where $C_i^\downarrow$ is obtained from C_i by replacing every variable X with the concept name A_X. It thus remains to show that the above $\mathcal{EL}^s$-concept is equivalent to $\nu_\ell X_1, \ldots, X_n.C_1, \ldots, C_n$ (in the above sense).

Let $\mathcal{I}$ be an interpretation, $\mathcal{V}$ an assigment, and $d \in (\nu_\ell X_1, \ldots, X_n.C_1, \ldots, C_n)^{\mathcal{I},\mathcal{V}}$. We obtain the interpretation $\mathcal{J}$ from $\mathcal{I}$ by setting $A_{X_i}^{\mathcal{J}} = (\nu_i X_1, \ldots, X_n.C_1, \ldots, C_n)^{\mathcal{I},\mathcal{V}}$ for $1 \leq i \leq n$ and $A_Y^{\mathcal{J}} = \mathcal{V}(Y)$ for all concept variables $Y \notin \{X_1, \ldots, X_n\}$. Thus, $d \in A_{X_\ell}^{\mathcal{J}}$. Since $\nu_i X_1 \cdots X_n.C_1, \ldots, C_n \equiv C_i(\nu_i X_1 \cdots X_n.C_1, \ldots, C_n)$ for $1 \leq i \leq n$ and by definition of $\mathcal{J}$, $\mathcal{J}$ satisfies $A_{X_i} \sqsubseteq C_i^\downarrow$ for $1 \leq i \leq n$. By definition of $\mathcal{J}$, the identity map on $\Delta^{\mathcal{I}}$ is an $\Sigma \setminus \{A_{X_1}, \ldots, A_{X_n}\}$-simulation from $(\mathcal{J}, d)$ to $(\mathcal{I}, d)$. It follows that $d \in \exists^{sim}\{A_{X_1}, \ldots, A_{X_n}\}.(\{A_{X_i} \sqsubseteq C_i^\downarrow \mid 1 \leq i \leq n\}, A_{X_\ell})$ as required.

Let $\mathcal{I}$ be an interpretation with $d \in (\exists^{sim}\{A_{X_1}, \ldots, A_{X_n}\}.(\{A_{X_i} \sqsubseteq C_i^\downarrow \mid 1 \leq i \leq n\}, A_{X_\ell})^{\mathcal{I}}$. Then there exists a model $\mathcal{J}$ of $\{A_{X_i} \sqsubseteq C_i^\downarrow \mid 1 \leq i \leq n\}$ and an $e \in A_{X_\ell}^{\mathcal{J}}$ such that $(\mathcal{J}, e) \leq_{\Sigma \setminus \{A_{X_1}, \ldots, A_{X_n}\}} (\mathcal{I}, d)$. Define an assignment $\mathcal{V}_{\mathcal{I}}$ by setting $\mathcal{V}_{\mathcal{I}}(X) = A_X^{\mathcal{I}}$ for all variables X, and similarly for $\mathcal{V}_{\mathcal{J}}$. We have to show that

$d \in (\nu_\ell X_1, \ldots, X_n.C_1, \ldots, C_n)^{\mathcal{I},\mathcal{V}_\mathcal{I}}$. To do this, we establish two claims.

Claim 1. For all ordinals α and $1 \leq i \leq n$, $\mathcal{V}_\mathcal{J}(X_i) \subseteq (\nu_i^\alpha X_1, \ldots, X_n.C_1, \ldots, C_n)^{\mathcal{J},\mathcal{V}_\mathcal{J}}$.

The induction start is trivial, hence it remains to consider the induction step and the transfinite step:

- $\mathcal{V}_\mathcal{J}(X_i) \subseteq (\nu_i^{\alpha+1} X_1, \ldots, X_n.C_1, \ldots, C_n)^{\mathcal{J},\mathcal{V}_\mathcal{J}}$.
  Let $e' \in \mathcal{V}_\mathcal{J}(X_i)$. As required for $\mathcal{J}$ we have $A_{X_j}^\mathcal{J} = \mathcal{V}_\mathcal{J}(X_j)$ f.a. $1 \leq j \leq n$. The induction hypothesis claims $\mathcal{V}_\mathcal{J}(X_j) \subseteq (\nu_j^\alpha X_1, \ldots, X_n.C_1, \ldots, C_n)^{\mathcal{J},\mathcal{V}_\mathcal{J}}$ for all $1 \leq j \leq n$. Hence $(C_i^\downarrow)^{\mathcal{J},\mathcal{V}_\mathcal{J}} \subseteq (C_i(\nu_i^\alpha X_1, \ldots, X_n.C_1, \ldots, C_n))^{\mathcal{J},\mathcal{V}_\mathcal{J}}$. Since $\mathcal{J}$ is a model of $A_{X_i} \sqsubseteq C_i^\downarrow$ it follows that

  $$e' \in (C_i(\nu_i^\alpha X_1, \ldots, X_n.C_1, \ldots, C_n))^{\mathcal{J},\mathcal{V}_\mathcal{J}} = (\nu_i^{\alpha+1} X_1, \ldots, X_n.C_1, \ldots, C_n)^{\mathcal{J},\mathcal{V}_\mathcal{J}}.$$

- $\mathcal{V}_\mathcal{J}(X_i) \subseteq (\nu_i^\lambda X_1, \ldots, X_n.C_1, \ldots, C_n)^{\mathcal{J},\mathcal{V}_\mathcal{J}}$, λ a limit ordinal.
  Let $e' \in \mathcal{V}_\mathcal{J}(X_i)$. By IH, $e' \in (\nu_i^\alpha X_1, \ldots, X_n.C_1, \ldots, C_n)^{\mathcal{J},\mathcal{V}_\mathcal{J}}$ for all $\alpha < \lambda$ and thus

  $$e' \in \bigcap_{\alpha<\lambda} (\nu_i^\alpha X_1, \ldots, X_n.C_1, \ldots, C_n)^{\mathcal{J},\mathcal{V}_\mathcal{J}} = (\nu_i^\lambda X_1, \ldots, X_n.C_1, \ldots, C_n)^{\mathcal{J},\mathcal{V}_\mathcal{J}}.$$

The second claim is also proved by transfinite induction on α.

Claim 2. For $1 \leq i \leq n$ we have: if $e' \in (\nu_i^\alpha X_1, \ldots, X_n.C_1, \ldots, C_n)^{\mathcal{J},\mathcal{V}_\mathcal{J}}$ and $(\mathcal{J}, e') \leq_{\Sigma \setminus \{A_{X_1}, \ldots, A_{X_n}\}} (\mathcal{I}, d')$ then $d' \in (\nu_i^\alpha X_1, \ldots, X_n.C_1, \ldots, C_n)^{\mathcal{I},\mathcal{V}_\mathcal{I}}$.

Again, the induction start is trivial and we concentrate on the induction step and transfinite step:

- $d' \in (\nu_i^{\alpha+1} X_1, \ldots, X_n.C_1, \ldots, C_n)^{\mathcal{I},\mathcal{V}_\mathcal{I}}$.
  Let $(r_1, e_{i_1}), \ldots, (r_m, e_{i_m})$ be those elements of $\mathsf{N_R} \times \Delta^\mathcal{J}$ such that $(e', e_{i_j}) \in r_j^\mathcal{I}$ for $1 \leq j \leq m$. Since $(\mathcal{J}, e') \leq (\mathcal{I}, d')$, we find $d_1, \ldots, d_m \in \Delta^\mathcal{I}$ such that $(d', d_j) \in r_j^\mathcal{I}$ and $(\mathcal{J}, e_{i_j}) \leq (\mathcal{I}, d_j)$ for $1 \leq j \leq m$. The induction hypothesis yields $d_j \in (\nu_{i_j}^\alpha d_1 \cdots d_n.C_1, \ldots, C_n)^{\mathcal{I},\mathcal{V}_\mathcal{I}}$. By definition of the $C_1, \ldots, C_n$ and the semantics of $\nu_i^{\alpha+1} d_1 \cdots d_n.C_1, \ldots, C_n$, this yields $d' \in (\nu_i^{\alpha+1} d_1 \cdots d_n.C_1, \ldots, C_n)^{\mathcal{I},\mathcal{V}_\mathcal{I}}$ as required.
- $d' \in (\nu_i^\lambda X_1, \ldots, X_n.C_1, \ldots, C_n)^{\mathcal{I},\mathcal{V}_\mathcal{I}}$, λ a limit ordinal.
  If $e' \in (\nu_i^\alpha X_1, \ldots, X_n.C_1, \ldots, C_n)^{\mathcal{I},\mathcal{V}_\mathcal{I}}$ for all $\alpha < \lambda$ then, by the induction hypothesis, $d' \in (\nu_i^\alpha X_1, \ldots, X_n.C_1, \ldots, C_n)^{\mathcal{I},\mathcal{V}_\mathcal{I}}$ for all $\alpha < \lambda$. Thus $d' \in (\nu_i^\lambda X_1, \ldots, X_n.C_1, \ldots, C_n)^{\mathcal{I},\mathcal{V}_\mathcal{I}}$.

It remains to argue that Claims 1 and 2 imply $d \in (\nu_\ell X_1, \ldots, X_n.C_1, \ldots, C_n)^{\mathcal{I},\mathcal{V}_\mathcal{I}}$: Since $e \in A_{X_\ell}^\mathcal{J}$, we have $e \in \mathcal{V}(X_\ell)$. By Claim 1, $e \in (\nu_\ell^\alpha X_1, \ldots, X_n.C_1, \ldots, C_n)^{\mathcal{J},\mathcal{V}_\mathcal{J}}$ for all ordinals α. By Claim 2, $d \in (\nu_\ell^\alpha X_1, \ldots, X_n.C_1, \ldots, C_n)^{\mathcal{J},\mathcal{V}_\mathcal{I}}$ for all ordinals α. Thus, $d \in (\nu_\ell X_1, \ldots, X_n.C_1, \ldots, C_n)^{\mathcal{I},\mathcal{V}_\mathcal{I}}$ as required.

We are now going to prove Theorem 4. First, we require the following [**?**] result.

Lemma 5. *It can be checked in poly-time whether there exists a (Σ-)simulation between two finite pointed interpretations.*

We give a slightly modified formulation of Theorem 4.

Theorem 9. *It is decidable in polynomial time whether an $\mathcal{EL}^{sa}$-KB is satisfiable. Moreover, given a satisfiable $\mathcal{EL}^{sa}$-KB $(\mathcal{T}, \mathcal{A})$, one can construct in polynomial time an interpretation $\mathcal{I}_{\mathcal{T},\mathcal{A}}$ with $|\Delta^{\mathcal{I}_{\mathcal{T},\mathcal{A}}}|$ bounded by twice the size of $(\mathcal{T}, \mathcal{A})$ and the following property:*

(CAN) $\mathcal{I}_{\mathcal{T},\mathcal{A}}$ is a model of $(\mathcal{T}, \mathcal{A})$ with $a^{\mathcal{I}_{\mathcal{T},\mathcal{A}}} = a$ for all $a \in \mathsf{Ind}(\mathcal{A})$ such that for every model $\mathcal{J}$ of $(\mathcal{T}, \mathcal{A})$ there exists a simulation S between $\mathcal{I}_{\mathcal{T},\mathcal{A}}$ and $\mathcal{J}$ with $(a, a^{\mathcal{J}}) \in S$ for all $a \in \mathsf{Ind}(\mathcal{A})$.

Proof. The construction of $\mathcal{I}_{\mathcal{T},\mathcal{A}}$ will be given by induction on the number of nestings of $\exists^{sim}$ in $\mathcal{T}$. For a $\mathcal{EL}^{sa}$-concept C, denote by $\mathsf{sub}_0(C)$ the set of subconcepts of C that are not within the scope of any $\exists^{sim}$. For a TBox $\mathcal{T}$ we set

$$\mathsf{sub}_0(\mathcal{T}) = \bigcup_{C \sqsubseteq D \in \mathcal{T}} \mathsf{sub}_0(C) \cup \mathsf{sub}_0(D).$$

Suppose $(\mathcal{T}, \mathcal{A})$ is given. By induction, we may assume that for any

$$F = \exists^{sim} \Sigma.(\mathcal{T}', \mathcal{A}', a') \in \mathsf{sub}_0(\mathcal{T})$$

we have decided already whether $(\mathcal{T}', \mathcal{A}')$ is satisfiable and, if so, have constructed an interpretation $\mathcal{I}_F = \mathcal{I}_{\mathcal{T}',\mathcal{A}'}$ satisfying (CAN). If F is not satisfiable, we replace F by $\bot$ everywhere in $\mathcal{T}$. Clearly the resulting TBox (denoted, for simplicity, by $\mathcal{T}$ as well) is equivalent to $\mathcal{T}$. If F is satisfiable, we construct an isomorphic copy $\mathcal{I}_F'$ of $\mathcal{I}_F$ with the following modifications:

- the domains $\Delta^{\mathcal{I}_F'}$ of $\mathcal{I}_F'$ are mutually disjoint and disjoint from $\mathsf{Ind}(\mathcal{A})$;
- individual names are not interpreted in $\mathcal{I}_F'$;
- all Σ-symbols are interpretated as empty sets;
- the point a' of $\mathcal{I}_F$ is renamed to $d_{\mathcal{I}_F}$ in $\mathcal{I}_F'$.

We can construct a model of $(\mathcal{T}, \mathcal{A})$ only if $(\mathcal{T}, \mathcal{A})$ is satisfiable. To enable us to construct a model that can be used to check satisfiability of $(\mathcal{T}, \mathcal{A})$, we first replace every occurrence of $\bot$ in $\mathcal{T}$ that is not within the scope of any $\exists^{sim}$ by a fresh concept names $A_\bot$ and denote the resulting TBox by $\mathcal{T}^\bot$. We construct the model $\mathcal{I}_{\mathcal{T}^\bot,\mathcal{A}}$ satisfying (CAN) for $(\mathcal{T}^\bot, \mathcal{A})$ and then decide, depending on the interpretation of $A_\bot$, whether be obtain a model of $(\mathcal{T}, \mathcal{A})$ or $(\mathcal{T}, \mathcal{A})$ is unsatisfiable. Let

$$\Delta_0 = \mathsf{Ind}(\mathcal{A}) \cup \{d_C \mid \exists r.C \in \mathsf{sub}_0(\mathcal{T}^\bot)\},$$

where the d_C are pairwise distinct fresh objects. We define an interpretation $\mathcal{I}_0$ as follows. Let

$$\Delta^{\mathcal{I}_0} = \Delta_0 \cup \bigcup_{F \in \mathsf{sub}_0(\mathcal{T}^\bot), F = \exists^{sim} \Sigma.(\mathcal{T}', \mathcal{A}', a')} \Delta^{\mathcal{I}_F'}$$

and set $(d_1, d_2) \in r^{\mathcal{I}_0}$ iff

1.0 $d_1 = a_1, d_2 = a_2$ and $r(a_1, a_2) \in \mathcal{A}$, or

1.1 $d_1 = d_{C_1} \in \Delta_0, d_2 = d_{C_2} \in \Delta_0$ and $\exists r.C_2$ is a top-tevel conjunct of C_1, or
1.2 $d_1, d_2 \in \Delta^{\mathcal{I}'_F}$ for some F and $(d_1, d_2) \in r^{\mathcal{I}'_F}$, or
1.3 $d_1 = d_{C_1}$, C_1 has a top-level conjunct $F = \exists^{sim}\Sigma.(\mathcal{T}', \mathcal{A}', a')$ and $(d_{\mathcal{I}_F}, d_2) \in r^{\mathcal{I}'_F}$.

Finally, set $d \in A^{\mathcal{I}_0}$ iff

2.0 $d = a_0$ and $A(a_0) \in \mathcal{A}$, or
2.1 $d = d_D \in \Delta_0$ and A is a top-level conjunct of D;
2.2 $d \in \Delta^{\mathcal{I}'_F}$ for some F and $d \in A^{\mathcal{I}'_F}$, or
2.3 $d = d_D$, D has a top-level conjunct $F = \exists^{sim}\Sigma.(\mathcal{T}', \mathcal{A}', a')$ and $d_{\mathcal{I}_F} \in A^{\mathcal{I}'_F}$.

It remains to satisfy the inclusions of $\mathcal{T}^{\perp}$ in $\mathcal{I}_0$. We may assume that, in each $C \sqsubseteq D \in \mathcal{T}^{\perp}$, D is a concept name, of the form $\exists r.D'$, or of the form $\exists^{sim}\Sigma.(\mathcal{T}', \mathcal{A}', a')$. Now we expand $\mathcal{I}_0$ by applying exhaustively the following rules:

3.1 Let $C \sqsubseteq A \in \mathcal{T}^{\perp}$ and assume that $d \in C^{\mathcal{I}_0}$ but $d \notin A^{\mathcal{I}_0}$. Then update $\mathcal{I}_0$ by setting $A^{\mathcal{I}_0} := \{d\} \cup A^{\mathcal{I}_0}$ (and leaving everything else unchanged).
3.2 Let $C \sqsubseteq \exists r.D \in \mathcal{T}^{\perp}$ and assume that $d \in C^{\mathcal{I}_0}$ but $d \notin (\exists r.D)^{\mathcal{I}_0}$. Then update $\mathcal{I}_0$ by setting $r^{\mathcal{I}_0} := \{(d, d_D)\} \cup r^{\mathcal{I}_0}$ (and leaving everything else unchanged).
3.3 Let $C \sqsubseteq F \in \mathcal{T}^{\perp}$ for $F = \exists^{sim}\Sigma.(\mathcal{T}', \mathcal{A}', a')$ and assume that $d \in C^{\mathcal{I}_0}$ but $d \notin F^{\mathcal{I}_0}$. Then update $\mathcal{I}_0$ by adding (d, d') to $r^{\mathcal{I}_0}$ whenever $(d_{\mathcal{I}_F}, d') \in r^{\mathcal{I}'_F}$ and by adding d to $A^{\mathcal{I}_0}$ whenever $d_{\mathcal{I}_F} \in A^{\mathcal{I}'_F}$.

The resulting interpretation is denoted by $\mathcal{I}_{\mathcal{T}^{\perp},\mathcal{A}}$. We show that $\mathcal{I}_{\mathcal{T}^{\perp},\mathcal{A}}$ satisfies (CAN) for $(\mathcal{T}^{\perp}, \mathcal{A})$, that the construction above is in poly-time, and that the domain of $\mathcal{I}_{\mathcal{T}^{bot},\mathcal{A}}$ is of linear size. The latter is easily proved and left to the reader. We now prove (CAN).

Claim 1. For every model $\mathcal{J}$ of $(\mathcal{T}^{\perp}, \mathcal{A})$ there exists a simulation S between $\mathcal{I}_{\mathcal{T}^{\perp},\mathcal{A}}$ and $\mathcal{J}$ with $(a, a^{\mathcal{J}}) \in S$ for all $a \in \mathsf{Ind}(\mathcal{A})$.

The proof is by induction on the number of nestings of $\exists^{sim}$ in $\mathcal{T}^{\perp}$. We consider the induction step from n to $n+1$. The case $n = 0$ is proved in the same way and left to the reader.

Assume Claim 1 has been proved for all $(\mathcal{T}', \mathcal{A}')$ with at most n nestings of $\exists^{sim}$ and let $\mathcal{T}^{\perp}$ have $n+1$ nestings of $\exists^{sim}$. Assume that $\mathcal{J}$ is a model of $(\mathcal{T}^{\perp}, \mathcal{A})$. We construct a simulation $S : \mathcal{I}_{\mathcal{T}^{\perp},\mathcal{A}} \leq \mathcal{J}$ with $(a, a^{\mathcal{J}}) \in S$ for all $a \in \mathsf{Ind}(\mathcal{A})$. By induction hypothesis, for every $F = \exists^{sim}\Sigma.(\mathcal{T}', \mathcal{A}', a') \in \mathsf{sub}_0(\mathcal{T}^{\perp})$, Claim 1 holds for $(\mathcal{T}', \mathcal{A}')$. So we have, in particular, $(\mathcal{I}'_F, d_{\mathcal{I}_F}) \leq (\mathcal{I}, a'^{\mathcal{I}})$ for every model $\mathcal{I}$ of $(\mathcal{T}', \mathcal{A}')$.

To construct S, we first set

$$S_0 := \{(d_C, e) \in \Delta_0 \times \Delta^{\mathcal{J}} \mid C \in \mathsf{sub}_0(\mathcal{T}^{\perp}), e \in C^{\mathcal{J}}\} \cup \{(a, a^{\mathcal{J}}) \mid a \in \mathsf{Ind}(\mathcal{A})\}.$$

By definition, for each $F = \exists^{sim}\Sigma.(\mathcal{T}', \mathcal{A}', a') \in \mathsf{sub}_0(\mathcal{T}^{\perp})$ and $e \in F^{\mathcal{J}}$ there exists a model $\mathcal{J}'$ of $(\mathcal{T}', \mathcal{A}')$ with $f = a'^{\mathcal{J}'}$ such that

$$(\mathcal{J}', f) \leq_{\Gamma} (\mathcal{J}, e),$$

where $\Gamma = (\mathsf{N_C} \cup \mathsf{N_R}) \setminus \Sigma$. By induction hypothesis, we have $(\mathcal{I}_{\mathcal{T}',\mathcal{A}'}, a') \leq (\mathcal{J}', f)$. Thus,

$$(\mathcal{I}'_F, d_{\mathcal{I}_F}) \leq (\mathcal{J}, e). \tag{1}$$

Let $S_{F,e}$ be a simulation that witnesses $(\mathcal{I}'_F, d_{\mathcal{I}_F}) \leq (\mathcal{J}, e)$ and let

$$S := S_0 \cup \bigcup \{S_{F,e} \mid e \in F^{\mathcal{J}}, F := \exists^{sim}\Sigma.(\mathcal{T}', \mathcal{A}', a') \in \mathsf{sub}_0(\mathcal{T}^{\perp})\}.$$

We prove that S is a simulation between $\mathcal{I}_{\mathcal{T},C}$ and $\mathcal{J}$. Then, by definition, Claim 1 is proved. To this end, we first prove the following claim.

Claim 1.a. S is a simulation between $\mathcal{I}_0$ and $\mathcal{J}$.

Let $(d_1, e_1) \in S$. First, we show that if $d_1 \in A^{\mathcal{I}_0}$, then $e_1 \in A^{\mathcal{J}}$. Let $d_1 \in A^{\mathcal{I}_0}$. If $d_1 \in \Delta_0$, then $(d_1, e_1) \in S_0$. Assume first $d_1 = d_C$ and $e_1 \in C^{\mathcal{J}}$ for some concept C. If $d_C \in A^{\mathcal{I}_0}$, then we have two cases:

2.1 A is a top-level conjunct of C. As $e_1 \in C^{\mathcal{J}}$, we have $e_1 \in A^{\mathcal{J}}$, as required.
2.3 C has the top-level conjunct $F := \exists^{sim}\Sigma.(\mathcal{T}', \mathcal{A}', a')$ and $d_{\mathcal{I}_F} \in A^{\mathcal{I}'_F}$. Then $e_1 \in F^{\mathcal{J}}$ and so, by (1), we obtain $e_1 \in A^{\mathcal{J}}$.

Assume now that $d_1 = a \in \mathsf{Ind}(\mathcal{A})$. Then $e_1 = a^{\mathcal{J}}$ and $A(a) \in \mathcal{A}$. We obtain $e_1 = a^{\mathcal{J}} \in A^{\mathcal{J}}$ from the condition that $\mathcal{J}$ is a model of $(\mathcal{T}, \mathcal{A})$.

In case $d_1 \in \Delta^{\mathcal{I}'_F}$ for some $F = \exists^{sim}\Sigma.(\mathcal{T}', \mathcal{A}', a') \in \mathsf{sub}_0(\mathcal{T}^{\perp})$ Item 2.2 applies. As $(d_1, e_1) \in S$ and $d_1 \in \Delta^{\mathcal{I}'_F}$ the definition of S stipulates $(d_1, e_1) \in S_{F,e}$ for some $e \in \Delta^{\mathcal{J}}$. Then $e_1 \in A^{\mathcal{J}}$ because $S_{F,e}$ is a simulation.

Let now $(d_1, d_2) \in r^{\mathcal{I}_{\mathcal{T},\mathcal{A}}}$. We have to show that there is $e_2 \in \Delta^{\mathcal{J}}$ such that $(e_1, e_2) \in r^{\mathcal{J}}$ and $(d_2, e_2) \in S$. We distinguish the following cases:

1.0 $d_1 = a_1$, $d_2 = a_2$ and $r(a_1, a_2) \in \mathcal{A}$. By definition of S, $e_1 = a_1^{\mathcal{J}}$. As $\mathcal{J}$ is a model of $(\mathcal{T}, \mathcal{A})$, we obtain $(e_1, e_2) \in r^{\mathcal{J}}$ for $e_2 = a_2^{\mathcal{J}}$. Moreover, $(d_2, e_2) \in S$.
1.1 Both, $d_1 = d_{C_1}$ and $d_2 = d_{C_2}$ are in Δ_0 and $\exists r.C_2$ a top-level conjunct of C_1. As $(d_{C_1}, e_1) \in S_0$ we have $e_1 \in C_1^{\mathcal{J}}$ and so there is $e_2 \in C_2^{\mathcal{J}}$ with $(e_1, e_2) \in r^{\mathcal{J}}$. Hence $(d_1, e_2) \in S$.
1.3 $d_1 = d_{C_1} \in \Delta^{\mathcal{I}_0}$ where $F := \exists^{sim}\Sigma(\mathcal{T}', \mathcal{A}', a')$ is a top-level conjunct of C_1, and $(d_{\mathcal{I}_F}, d_2) \in r^{\mathcal{I}'_F}$. Since $e_1 \in C_1^{\mathcal{J}}$ we have $e_1 \in F^{\mathcal{J}}$. By (1), $(\mathcal{I}'_F, d_{\mathcal{I}_F}) \leq (\mathcal{J}, e_1)$. Thus, there exists $e_2 \in \Delta^{\mathcal{J}}$ with $(e_1, e_2) \in r^{\mathcal{J}}$ and $(d_2, e_2) \in S_{F,e_1}$. We obtain $(d_2, e_2) \in S$.
1.2 Both d_1 and d_2 are in $\Delta^{\mathcal{I}'_F}$. As we assume $(d_1, e_1) \in S$ there must be $S_{F,e'} \subseteq S$ containing (d_1, e_1). We have $(d_1, d_2) \in r^{\mathcal{I}'_F}$ and so, since $S_{F,e'}$ is a simulation, there exists $e_2 \in \Delta^{\mathcal{J}}$ with $(e_1, e_2) \in r^{\mathcal{J}}$ such that $(d_2, e_2) \in S_{F,e'}$. Hence $(d_2, e_2) \in S$.

Claim 1.a. is proved. To prove Claim 1, it remains to show that iterated applications of the rules (3.1)-(3.3) to $\mathcal{I}_0$ preserve the condition that S is a simulation. So, assume that $\mathcal{I}_0 \subseteq \mathcal{I}_1 \subseteq \ldots$ is a sequence resulting from applications of the rules (3.1)-(3.3) and assume that S is a simulation between $\mathcal{I}_k$ and $\mathcal{J}$ such that $(a, a^{\mathcal{J}}) \in S$ for all $a \in \mathsf{Ind}(\mathcal{A})$. We show S is a simulation between $\mathcal{I}_{k+1}$ and $\mathcal{J}$.

3.1 Assume $d_1 \in C^{\mathcal{I}_k}$, $C \sqsubseteq A \in \mathcal{T}$, and $\mathcal{I}_{k+1}$ coincides with $\mathcal{I}_k$ except that $A^{\mathcal{I}_{k+1}} := A^{\mathcal{I}_k} \cup \{d_1\})$. Clearly it is sufficient to show that $e_1 \in A^{\mathcal{J}}$ for every $(d_1, e_1) \in S$. We have $S : (\mathcal{I}_k, d_1) \leq (\mathcal{J}, e_1)$ and so $e_1 \in C^{\mathcal{J}}$. Hence, since $\mathcal{J} \models \mathcal{T}$, $e_1 \in A^{\mathcal{J}}$, as required.

3.2 Assume $d_1 \in C^{\mathcal{I}_k}$, $C \sqsubseteq \exists r.D \in \mathcal{T}$, and $\mathcal{I}_{k+1}$ coincides $\mathcal{I}_k$ except that $r^{\mathcal{I}_k} := r^{\mathcal{I}_k} \cup \{(d_1, d_D)\})$. Assume that $(d_1, e_1) \in S$ for some $e_1 \in \Delta^{\mathcal{J}}$. Clearly it is sufficient to show that there exists e_2 with $(e_1, e_2) \in r^{\mathcal{J}}$ such that $(d_D, e_2) \in S$. We have $e_1 \in C^{\mathcal{J}}$ and as $\mathcal{J} \models \mathcal{T}$ by assumption there is an $e_2 \in D^{\mathcal{J}}$ with $(e_1, e_2) \in r^{\mathcal{J}}$. According to the definition of S we have $(d_D, e_2) \in S$, as required.

3.3 Assume $d_1 \in C^{\mathcal{I}_k}$, $C \sqsubseteq F \in \mathcal{T}$ with $F = \exists^{sim}\Sigma.(\mathcal{T}', \mathcal{A}', a')$ and $\mathcal{I}_{k+1}$ coincides with $\mathcal{I}_k$ except that

$$r^{\mathcal{I}_{k+1}} = r^{\mathcal{I}_k} \cup \{(d_1, d_2) \mid (d_{\mathcal{I}'_F}, d_2) \in r^{\mathcal{I}'_F}\}, \quad A^{\mathcal{I}_{k+1}} = r^{\mathcal{I}_k} \cup \{d_1 \mid d_{\mathcal{I}'_F} \in A^{\mathcal{I}'_F}\}$$

for $r \in \mathsf{N_R}$ and $A \in \mathsf{N_C}$. We consider the condition on roles. Let $(d_{\mathcal{I}'_F}, d_2) \in r^{\mathcal{I}'_F}$ and let $(d_1, e_1) \in S$ for some $e_1 \in \Delta^{\mathcal{J}}$. Clearly it is enough to show that there exists e_2 with $(e_1, e_2) \in r^{\mathcal{J}}$ and $(d_2, e_2) \in S$. We have $e_1 \in C^{\mathcal{J}}$ and as $\mathcal{J} \models \mathcal{T}$ by assumption, we have $e_1 \in F^{\mathcal{J}}$. So $S_{F,e_1} \subseteq S$ and we have $S_{F,e_1} : (\mathcal{I}'_F, d_{\mathcal{I}_F}) \leq (\mathcal{J}, e_1)$. Hence there exists e_2 with $(e_1, e_2) \in r^{\mathcal{J}}$ and $(d_2, e_2) \in S_{F,e_1}$, as required.

This finishes the proof of Claim 1. We now show

Claim 2. The following conditions hold.

– $\mathcal{I}_{\mathcal{T}^\perp,\mathcal{A}}$ is a model of $(\mathcal{T}^\perp \mathcal{A})$;
– for all $a \in \mathsf{Ind}(\mathcal{A})$ and all D: $(\mathcal{T}^\perp, \mathcal{A}) \models D(a)$ iff $a \in D^{\mathcal{I}_{\mathcal{T}^\perp,\mathcal{A}}}$;
– for all $d_C \in \Delta_0$ and all D: $(\mathcal{T}^\perp, \mathcal{A}) \models C \sqsubseteq D$ iff $d_C \in D^{\mathcal{I}_{\mathcal{T}^\perp,\mathcal{A}}}$.

The proof is again by induction on the number of nestings of $\exists^{sim}$. Assume this has been proved for all $\mathcal{T}', \mathcal{A}'$ such that $\exists^{sim}\Sigma.(\mathcal{T}', \mathcal{A}', a')$ occurs in $\mathcal{T}^\perp$.

For Point 1 note that $\mathcal{I}_{\mathcal{T}^\perp,\mathcal{A}}$ is, by definition, a model of $\mathcal{A}$. $\mathcal{I}_{\mathcal{T}^\perp,\mathcal{A}} \models \mathcal{T}$ follows immediately from the fact that none of the rules (3.1)-(3.3) is applicable to $\mathcal{I}_{\mathcal{T}^\perp,\mathcal{A}}$ and that rule (3.3) constructs a model satisfying the required $F = \exists^{sim}\Sigma.(\mathcal{T}', \mathcal{A}', a')$ because $\mathcal{I}'_F$ is the reduct of the interpretation $\mathcal{I}_{\mathcal{T}',\mathcal{A}'}$ not interpreting Σ-symbols that is, by induction hypothesis, a model of $(\mathcal{T}', \mathcal{A}')$. Points 2 and 3 follow immediately from the induction hypothesis, Claim 1, and Point 1.

It follows that $\mathcal{I}_{\mathcal{T}^\perp,\mathcal{A}}$ satisfies (CAN) for $(\mathcal{T}^\perp, \mathcal{A})$. We now use this model to check satisfiability of $(\mathcal{T}, \mathcal{A})$ and construct $\mathcal{I}_{\mathcal{T},\mathcal{A}}$ in case it is satisfiable. Denote by $\mathcal{I}_{\mathcal{T},\mathcal{A}}$ the restriction of $\mathcal{I}_{\mathcal{T}^\perp,\mathcal{A}}$ to all $d \in \Delta^{\mathcal{I}_{\mathcal{T}^\perp,\mathcal{A}}}$ such that there exist $a \in \mathsf{Ind}(\mathcal{A})$, $d_0, \ldots, d_n = d$, and role names $r_1, \ldots, r_n$ with $d_0 = a$ and $(d_i, d_{i+1}) \in r_{i+1}^{\mathcal{I}_{\mathcal{T}^\perp,\mathcal{A}}}$ for $i < n$.

Claim 3. $(\mathcal{T}, \mathcal{A})$ is satisfiable iff $A_\perp^{\mathcal{I}_{\mathcal{T},\mathcal{A}}} = \emptyset$. Moreover, if $A_\perp^{\mathcal{I}_{\mathcal{T},\mathcal{A}}} = \emptyset$, then $\mathcal{I}_{\mathcal{T},\mathcal{A}}$ satisfies (CAN).

Clearly, $\mathcal{I}_{\mathcal{T},\mathcal{A}}$ still has the properties from Claim 1 and 2. Moreover, for $d \in \Delta^{\mathcal{I}_{\mathcal{T},\mathcal{A}}}$, we have $d \in A_\perp^{\mathcal{I}_{\mathcal{T},\mathcal{A}}}$ iff there exists $a \in \mathsf{Ind}(\mathcal{A})$ such that

$$(\mathcal{T}^\perp, \mathcal{A}) \models \exists r_1.\exists r_2.\cdots\exists r_n.A_\perp(a)$$

for some sequence $r_1, \ldots, r_n$ of role names. It thus follows from the construction of $\mathcal{T}^\perp$ from $\mathcal{T}$ that $(\mathcal{T}, \mathcal{A})$ is satisfiable iff $A_\perp^{\mathcal{I}_{\mathcal{T},\mathcal{A}}} = \emptyset$. The proof of the second claim is straightforward and left to the reader.

Finally, we show that $\mathcal{I}_{\mathcal{T},\mathcal{A}}$ can be constructed in polynomial time. As the domain of $\mathcal{I}_{\mathcal{T},\mathcal{A}}$ is clearly of linear size, it is sufficient to show that one can check the pre-condition "$d \in C^{\mathcal{I}}$" of the rules (3.1)-(3.3) in polynomial time. The only concepts C of interest are of the form $\exists^{sim}\Sigma.(\mathcal{T}, \mathcal{A}, a)$. But we know that checking $d \in (\exists^{sim}\Sigma.(\mathcal{T}, \mathcal{A}, a))^{\mathcal{I}}$ is equivalent to checking whether $(\mathcal{T}, \mathcal{A})$ is satisfiable and there exists a Γ-simulation between between $(\mathcal{I}_{\mathcal{T},\mathcal{A}}, a)$ and $(\mathcal{I}, d)$, for $\Gamma = (\mathsf{N_C} \cup \mathsf{N_I}) \setminus \Sigma$. Thus, checking this pre-condition is in polynomial time, by Lemma 5.

C Proofs for Section 6

To prove Theorem 8, we require canonical models for TBoxes and concepts instead of KBs. For a $\mathcal{EL}^\nu$-TBox $\mathcal{T}$ and $\mathcal{EL}^\nu$-concept C we denote by $(\mathcal{I}_{\mathcal{T},C}, d_C)$ a pointed interpretation such that

- $(\mathcal{I}_{\mathcal{T},C}, d_C)$ is a tree-model of finite outdegree and root d_C;
- for every model $\mathcal{J}$ of $\mathcal{T}$ with $e \in C^{\mathcal{J}}$, we have $(\mathcal{I}_{\mathcal{T},C}, d_C) \leq (\mathcal{J}, e)$;
- $\mathcal{I}_{\mathcal{T},C}$ is a model of $\mathcal{T}$ such that $d_C \in C^{\mathcal{I}_{\mathcal{T},C}}$.

$\mathcal{I}_{\mathcal{T},C}$ can be obtained as the tree-unraveling (introduced above) of $\mathcal{I}_{\mathcal{T}',\mathcal{A},a}$ at a, where $\mathcal{T}' = \mathcal{T} \cup \{A \sqsubseteq C\}$ and $\mathcal{A} = \{A(a)\}$ for a fresh concept name A. We formulate the result to be proved again.

Theorem 8 Let $\mathcal{T}_1 \cup \mathcal{T}_2 \models C_0 \sqsubseteq D_0$ and assume that $\mathcal{T}_1$ and $\mathcal{T}_2$ are Γ-inseparable w.r.t. $\mathcal{EL}^\nu$ for $\Gamma = \mathsf{sig}(\mathcal{T}_1, C_0) \cap \mathsf{sig}(\mathcal{T}_2, D_0)$. Then $\mathcal{T}_2 \models \exists^{sim}\Sigma.(\mathcal{T}_1, C_0) \sqsubseteq D_)$, for $\Sigma = \mathsf{sig}(\mathcal{T}_1, C_0) \setminus \Gamma$.

Proof. Assume that $\mathcal{T}_2 \not\models \exists^{sim}\Sigma.(\mathcal{T}_1, C_0) \sqsubseteq D_0$, for $\Sigma = \mathsf{sig}(\mathcal{T}_1, C_0) \setminus \Delta$. We show that $\mathcal{T}_1 \cup \mathcal{T}_2 \not\models C_0 \sqsubseteq D_0$.

Take the canonical model $(\mathcal{I}_{\mathcal{T}_1,C_0}, d_{C_0})$ defined above. We set $\mathcal{I}_0 = \mathcal{I}_{\mathcal{T}_1,C_0}$, $d_0 = d_{C_0}$, and $\Delta_0 = \Delta_{d_0} = \Delta^{\mathcal{I}_0}$. In the following, we construct an interpretation $\mathcal{I}^*$ of $\mathcal{T}_1 \cup \mathcal{T}_2$ refuting $C_0 \sqsubseteq D_0$. We define inductively an infinite sequence $\mathcal{I}_1, \mathcal{I}_2, \ldots$ of interpretations. The interpretation $\mathcal{I}^* = (\Delta^{\mathcal{I}^*}, \cdot^{\mathcal{I}^*})$ is then defined as the union of $\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2, \ldots$ as follows:

$$\Delta^{\mathcal{I}^*} := \bigcup_{i \geq 0} \Delta^{\mathcal{I}_i};$$
$$A^{\mathcal{I}^*} := \bigcup_{i \geq 0} A^{\mathcal{I}_i}, \text{ for all } A \in \mathsf{N_C};$$
$$r^{\mathcal{I}^*} := \bigcup_{i \geq 0} r^{\mathcal{I}_i}, \text{ for all } r \in \mathsf{N_R}.$$

Given an intepretation $\mathcal{I}$, we denote by $\mathcal{I} \restriction \Gamma$ the reduct of $\mathcal{I}$ interpreting the symbols in Γ only. For $d \in \Delta^{\mathcal{I}}$ and any TBox $\mathcal{T}$, we denote by $\mathcal{I}_{t_{\mathcal{I}}(d),\mathcal{T}}$ the canonical model $\mathcal{I}_{\mathcal{T},\exists^{sim}(\mathcal{I} \restriction \Gamma, d)}$ of the pair $\mathcal{T}, \exists^{sim}(\mathcal{I} \restriction \Gamma, d)$.

Let $n \geq 0$ and assume the interpretation $\mathcal{I}_n$ with domain Δ_n has been defined. If n is even, then take for every $d \in \Delta_n \setminus \Delta_{n-1}$ (we set $\Delta_{-1} = \emptyset$) the interpretation $\mathcal{I}_d = \mathcal{I}_{t_{\mathcal{I}_n}(d),\mathcal{T}_2}$ with domain Δ_d such that $\Delta_n \cap \Delta_d = \{d\}$ and the Δ_d, $d \in \Delta_n \setminus \Delta_{n-1}$, are mutually disjoint. If n is odd, then take for every $d \in \Delta_n \setminus \Delta_{n-1}$ the interpretation $\mathcal{I}_d = \mathcal{I}_{t_{\mathcal{I}_n}(d),\mathcal{T}_1}$ with domain Δ_d such that $\Delta_n \cap \Delta_d = \{d\}$ and the Δ_d, $d \in \Delta_n \setminus \Delta_{n-1}$, are mutually disjoint. Now set

$$\begin{aligned} \Delta_{n+1} &= \Delta_n \cup \textstyle\bigcup_{d \in \Delta_n \setminus \Delta_{n-1}} \Delta_d, \\ r^{\mathcal{I}_{n+1}} &= r^{\mathcal{I}_n} \cup \textstyle\bigcup_{d \in \Delta_n \setminus \Delta_{n-1}} r^{\mathcal{I}_d}, \\ A^{\mathcal{I}_{n+1}} &= A^{\mathcal{I}_n} \cup \textstyle\bigcup_{d \in \Delta_n \setminus \Delta_{n-1}} A^{\mathcal{I}_d}. \end{aligned}$$

For all $d \in \Delta^{\mathcal{I}^*}$ there exists a (uniquely) determined minimal natural number $n(d)$ with $d \in \Delta_{n(d)} \setminus \Delta_{n(d)-1}$. If $n(d) \neq 0$, then there exists a uniquely determined $d^* \in \Delta_{n(d)-1}$ with $d \in \Delta_{d^*}$. We set $d^* = d_0$ for $n(d) = 0$ and prove the following by induction on the construction of D. For all $d \in \Delta^{\mathcal{I}^*}$ and $\mathcal{EL}$-concepts D:

- if $n(d)$ is even then
  1. if $\mathsf{sig}(D) \cap \mathsf{sig}(\mathcal{T}_1, C_0) \subseteq \Gamma$, then $d \in D^{\mathcal{I}^*} \Leftrightarrow d \in D^{\mathcal{I}_d}$;
  2. if $\mathsf{sig}(D) \cap \mathsf{sig}(\mathcal{T}_2, D_0) \subseteq \Gamma$, then $d \in D^{\mathcal{I}^*} \Leftrightarrow d \in D^{\mathcal{I}_{d^*}}$;
- if $n(d)$ is odd then
  1. if $\mathsf{sig}(D) \cap \mathsf{sig}(\mathcal{T}_2, D_0) \subseteq \Gamma$, then $d \in D^{\mathcal{I}^*} \Leftrightarrow d \in D^{\mathcal{I}_d}$;
  2. if $\mathsf{sig}(D) \cap \mathsf{sig}(\mathcal{T}_1, C_0) \subseteq \Gamma$, then $d \in D^{\mathcal{I}^*} \Leftrightarrow d \in D^{\mathcal{I}_{d^*}}$.

The implications from right to left are trivial, so we consider the implications from left to right only. We concentrate on the case $n(d)$ even (the case $n(d)$ odd is proved in the same way) and prove the induction step for $D = \exists r.C$. First consider Point 1. So let $\mathsf{sig}(D) \cap \mathsf{sig}(\mathcal{T}_1, C_0) \subseteq \Gamma$ and assume $d \in D^{\mathcal{I}^*}$ with $n(d)$ even. There exists $c \in \Delta^{\mathcal{I}^*}$ such that $c \in C^{\mathcal{I}^*}$ and $(d, c) \in r^{\mathcal{I}^*}$. Assume first that $c \in \Delta_{n(d)}$. Then, by construction, $c \notin \Delta_{n(d)-1}$. Then $r \in \Gamma$ because for any $r \notin \mathsf{sig}(\mathcal{T}_1, C_0)$, $r^{\mathcal{I}^*} \cap (\Delta_{n(d)} \setminus \Delta_{n(d)-1})^2 = \emptyset$. We obtain $n(c) = n(d)$ and, by IH, $c \in C^{\mathcal{I}_c}$. We obtain, by the construction of $\mathcal{I}_c$,

$$\mathcal{T}_2 \models \exists^{sim}(\mathcal{I}_{n(d)}{\restriction}\Gamma, c) \sqsubseteq C$$

But then

$$\mathcal{T}_2 \models \exists r.\exists^{sim}(\mathcal{I}_{n(d)}{\restriction}\Gamma, c) \sqsubseteq \exists r.C$$

and so from the validity of

$$\exists^{sim}(\mathcal{I}_{n(d)}{\restriction}\Gamma, d) \sqsubseteq \exists r.\exists^{sim}(\mathcal{I}_{n(d)}{\restriction}\Gamma, c)$$

we obtain

$$\mathcal{T}_2 \models \exists^{sim}(\mathcal{I}_{n(d)}{\restriction}\Gamma, d) \sqsubseteq D$$

which implies $d \in D^{\mathcal{I}_d}$, as required.

Now assume $c \notin \Delta_{n(d)}$. Then $c \in \Delta_d$, $c^* = d$, and $n(c) = n(d) + 1$. By IH (for $n(c)$ odd), $c \in C^{\mathcal{I}^*}$ iff $c \in C^{\mathcal{I}_{c^*}} = C^{\mathcal{I}_d}$. Hence $d \in (\exists r.C)^{\mathcal{I}_d}$.

Consider now Point 2. Let $\mathsf{sig}(D) \cap \mathsf{sig}(\mathcal{T}_2, D_0) \subseteq \Gamma$ and $d \in D^{\mathcal{I}^*}$. There exists $c \in \Delta^{\mathcal{I}^*}$ such that $c \in C^{\mathcal{I}^*}$ and $(d, c) \in r^{\mathcal{I}^*}$. Assume first that $c \in \Delta_{d^*}$. Then $c^* = d^*$ and, by IH, $c \in C^{\mathcal{I}_{d^*}}$. As we also have $(d, c) \in r^{\mathcal{I}_{d^*}}$, we obtain $d \in D^{\mathcal{I}_{d^*}}$.

Now assume $c \notin \Delta_{d^*}$. Then $c \in \Delta_d$. Then $r \in \Gamma$ because for any $r \notin \mathsf{sig}(\mathcal{T}_2, D_0)$, $r^{\mathcal{I}^*} \cap \Delta_d \times \Delta_d = \emptyset$. By IH, $c \in C^{\mathcal{I}_c}$. Hence

$$\mathcal{T}_1 \models \exists^{sim}(\mathcal{I}_{n(d)+1} \restriction \Gamma, c) \sqsubseteq C.$$

Then

$$\mathcal{T}_1 \models \exists r.\exists^{sim}(\mathcal{I}_{n(d)+1} \restriction \Gamma, c) \sqsubseteq \exists r.C.$$

We have $d \in (\exists r.\exists^{sim}(\mathcal{I}_{n(d)+1} \restriction \Gamma, c))^{\mathcal{I}_d}$ and by Γ-inseparability of $\mathcal{T}_1$ and $\mathcal{T}_2$, $d \in (\exists r.\exists^{sim}(\mathcal{I}_{n(d)+1} \restriction \Gamma, c))^{\mathcal{I}_{n(d)}}$. So, $d \in (\exists r.\exists^{sim}(\mathcal{I}_{n(d)+1} \restriction \Gamma, c))^{\mathcal{I}_{d^*}}$. $\mathcal{I}_{d^*}$ is a model of $\mathcal{T}_1$. Hence $d \in (\exists r.C)^{\mathcal{I}_{d^*}}$, as required.

Since $\mathcal{I}^*$ has finite outdegree, it follows that the claim above holds for all $\mathcal{EL}^\nu$-concepts D (not just all $\mathcal{EL}$-concepts).

It follows immediately that $\mathcal{I}^*$ is a model of $\mathcal{T}_1 \cup \mathcal{T}_2$: let $C \sqsubseteq D \in \mathcal{T}_i$. If $C_0^{\mathcal{I}^*} \setminus D_0^{\mathcal{I}^*} \neq \emptyset$, then there exists a an interpretation $\mathcal{I}_d$ of $\mathcal{T}_i$ with $C_0^{\mathcal{I}_d} \setminus D_0^{\mathcal{I}_d} \neq \emptyset$ which contradicts the claim above.

It remains to show that $d_0 \in C_0^{\mathcal{I}^*} \setminus D_0^{\mathcal{I}^*}$. $d_0 \in C_0^{\mathcal{I}^*}$ is clear by definition. Now assume $d_0 \in D_0^{\mathcal{I}^*}$. By the claim above (for $\mathcal{EL}^\nu$-concepts), we obtain $d_0 \in D_0^{\mathcal{I}_d}$. By construction of $\mathcal{I}_{d_0}$, this implies

$$\mathcal{T}_2 \models \exists^{sim}(\mathcal{I}_{\mathcal{T}_1, C_0} \restriction \Gamma, d_0) \sqsubseteq D_0.$$

But the latter statement is equivalent to $\mathcal{T}_2 \models \exists^{sim}\Sigma.(\mathcal{T}_1, C_0) \sqsubseteq D_0$ and we have derived a contradiction.