Modularity in DL-Lite

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Abstract. We develop a formal framework for modular ontologies by analysing four notions of conservative extensions and their applications in refining, re-using, merging, and segmenting ontologies. For two members of the *DL-Lite* family of description logics, we prove important meta-properties of these notions such as robustness under joins, vocabulary extensions, and iterated import of ontologies. The computational complexity of the corresponding reasoning tasks is investigated.

1 Introduction

In computer science and related areas, ontologies are used to define the meaning of vocabularies designed to speak about some domains of interest. In ontology languages based on description logics (DLs), such an ontology typically consists of a TBox stating which inclusions hold between complex concepts built over the vocabulary. An increasingly important application of ontologies is management of large amounts of data, where the ontology is used to provide flexible and efficient access to repositories consisting of data sets of instances of concepts and relations of the vocabulary. In DLs, such repositories are typically modelled as ABoxes.

Developing ontologies for this and other purposes is a difficult task. When dealing with DLs, the ontology designer is supported by efficient reasoning tools for classification, instance checking and some other reasoning problems. However, it is generally recognised that this support is not sufficient when ontologies are not developed as 'monolithic entities' but rather result from importing, merging, combining, re-using, refining and extending existing ontologies. In all those cases, reasoning support for analysing the impact of the respective operation on the ontology would be highly desirable. Typical reasoning tasks in this case may include the following:

- If we add some new concepts, relations and axioms to our ontology, can new assertions over the vocabulary of the original TBox be derived from the extended TBox?
- When importing an ontology, do we change the meaning of its vocabulary?
- When looking for a definition of some concepts, what part of the existing ontology defining them should be used?

Recently, the notion of conservative extension has been identified as fundamental for dealing with problems of this kind [1–5]. Parameterising this notion by a language $\mathcal{L}$, we say that a TBox $\mathcal{T}$ is a *conservative extension* of a TBox $\mathcal{T}'$ w.r.t. $\mathcal{L}$ if $\mathcal{T} \models \alpha$ implies $\mathcal{T}' \models \alpha$, for every α from $\mathcal{L}$ which only uses the vocabulary of $\mathcal{T}'$. In these papers, the main emphasis has been on languages $\mathcal{L}$ consisting of TBox axioms over some description logic (such as $\mathcal{ALC}$) and the much stronger notion of *model conservativity* which corresponds to the assumption that α can be taken from any language with standard Tarski semantics (e.g., second-order logic). Considering TBox axioms is motivated by the fact that ontologies are developed and represented via such axioms. They are the syntactic objects an ontology designer is working with, and a possibility to derive some new axioms appears therefore to be a good indicator as to whether the meaning of symbols has changed in any relevant sense. The notion of model conservativity is motivated by its flexibility: whatever language $\mathcal{L}$ is chosen, no new consequences in $\mathcal{L}$ will be derivable [5, 4]. A third option (which lies between the two above as far as expressivity is concerned) is as follows: if the main application of the ontologies $\mathcal{T}$ and $\mathcal{T}'$ is to provide a vocabulary and its meaning for posing queries to ABoxes, then it appears to be of interest to regard $\mathcal{T}$ as a conservative extension of $\mathcal{T}'$ if, for every ABox $\mathcal{A}$ and every (say, positive existential) query q in the vocabulary of $\mathcal{T}'$, any answer to q given by $(\mathcal{T}, \mathcal{A})$ is given by $(\mathcal{T}', \mathcal{A})$ as well. It can thus be seen that there is a variety of notions of conservativity which can be used to formally define modularity in ontologies. The choice of the appropriate one depends on what the ontologies are supposed to be used for, the computational complexity of the corresponding reasoning tasks, and the relevant meta-properties and 'robustness' of the notion of conservativity.

Here we investigate these and related notions of conservative extensions for the *DL-Lite* family of description logics [6–8]. *DL-Lite* and its variants are weak descriptions logics that have been designed in order to facilitate efficient query-answering over large data sets. We introduce four different notions of conservativity for two languages within this family, motivate their relevance for modularity and re-use of ontologies, study their meta-properties, and determine the computational complexity of the corresponding reasoning tasks. All the proofs can be found in the Appendix available at `http://www.csc.liv.ac.uk/~frank`.

2 The *DL-Lite* Family

The *DL-Lite* family of DLs has been introduced and investigated in [6–8] with the aim of establishing maximal subsets of DL constructors for which the data complexity of query answering stays within LogSpace. The 'covering' DL of the *DL-Lite* family is known as $\textit{DL-Lite}_{bool}$ [8]. As $\textit{DL-Lite}_{bool}$ itself contains classical propositional logic, query answering in it is coNP-hard, but by taking the Horn-fragment $\textit{DL-Lite}_{horn}$ of $\textit{DL-Lite}_{bool}$, one obtains a language for which query answering is within LogSpace [8] (precise formulations of these results are given below).

The language of *DL-Lite*$_{bool}$ has *object names* $a_1, a_2, \ldots$, *concept names* $A_1, A_2, \ldots$, and *role names* $P_1, P_2, \ldots$. Complex *roles* R and *DL-Lite*$_{bool}$ *concepts* C are defined as follows:

$$
\begin{array}{lcl}
R & ::= & P_i \quad | \quad P_i^-, \\
B & ::= & \bot \quad | \quad \top \quad | \quad A_i \quad | \quad \geq q\, R, \\
C & ::= & B \quad | \quad \neg C \quad | \quad C_1 \sqcap C_2,
\end{array}
$$

where $q \geq 1$. The concepts of the form B above are called *basic*. A *DL-Lite*$_{bool}$ *concept inclusion* is of the form $C_1 \sqsubseteq C_2$, where C_1 and C_2 are *DL-Lite*$_{bool}$ concepts. A *DL-Lite*$_{bool}$ *TBox* is a finite set of *DL-Lite*$_{bool}$ concept inclusions. (Other concept constructs like $\exists R$, $\leq q\, R$ and $C_1 \sqcup C_2$ will be used as standard abbreviations.)

As mentioned above, we also consider the *Horn* fragment *DL-Lite*$_{horn}$ of *DL-Lite*$_{bool}$: a *DL-Lite*$_{horn}$ *concept inclusion* is of the form

$$\bigsqcap_k B_k \sqsubseteq B,$$

where B and the B_k are basic concepts. In this context, basic concepts will also be called *DL-Lite*$_{horn}$ *concepts*. Note that the axioms $\bigsqcap_k B_k \sqsubseteq \bot$ and $\top \sqsubseteq B$ are legal in *DL-Lite*$_{horn}$. A *DL-Lite*$_{horn}$ *TBox* is a finite set of *DL-Lite*$_{horn}$ concept inclusions. For other fragments of *DL-Lite*$_{bool}$ we refer the reader to [6–8]. It is worth noting that in *DL-Lite*$_{horn}$ we can express both *global functionality* of a role and *local functionality* (i.e., functionality restricted to a (basic) concept B) by means of the axioms $\geq 2\, R \sqsubseteq \bot$ and $B \sqcap \geq 2\, R \sqsubseteq \bot$, respectively.

Let $\mathcal{L}$ be either *DL-Lite*$_{bool}$ or *DL-Lite*$_{horn}$. An $\mathcal{L}$-*ABox* is a set of assertions of the form $C(a_i)$, $R(a_i, a_j)$, where each C is an $\mathcal{L}$-concept, R a role, and a_i, a_j are object names. An $\mathcal{L}$ *knowledge base* ($\mathcal{L}$-*KB*) is a pair $(\mathcal{T}, \mathcal{A})$ consisting of an $\mathcal{L}$-TBox $\mathcal{T}$ and an $\mathcal{L}$-ABox $\mathcal{A}$.

An *interpretation* $\mathcal{I}$ is a structure of the form $(\Delta^{\mathcal{I}}, A_1^{\mathcal{I}}, \ldots, P_1^{\mathcal{I}}, \ldots a_1^{\mathcal{I}}, \ldots)$, where $\Delta^{\mathcal{I}}$ is non-empty, $A_i^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$, $P_i^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ and $a_i^{\mathcal{I}} \in \Delta^{\mathcal{I}}$ such that $a_i^{\mathcal{I}} \neq a_j^{\mathcal{I}}$, for $a_i \neq a_j$ (i.e., we adopt the unique name assumption). The *extension* $C^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$ of a concept C is defined as usual. A concept inclusion $C_1 \sqsubseteq C_2$ is *satisfied* in $\mathcal{I}$ if $C_1^{\mathcal{I}} \subseteq C_2^{\mathcal{I}}$; in this case we write $\mathcal{I} \models C_1 \sqsubseteq C_2$. $\mathcal{I}$ is a *model for* a TBox $\mathcal{T}$ if all concept inclusions from $\mathcal{T}$ are satisfied in $\mathcal{I}$. A concept inclusions $C_1 \sqsubseteq C_2$ *follows from* $\mathcal{T}$, $\mathcal{T} \models C_1 \sqsubseteq C_2$ in symbols, if every model for $\mathcal{T}$ satisfies $C_1 \sqsubseteq C_2$. A concept C is $\mathcal{T}$-*satisfiable* if there exists a model $\mathcal{I}$ for $\mathcal{T}$ with $C^{\mathcal{I}} \neq \emptyset$. We say that $\mathcal{I}$ is a *model* for an $\mathcal{L}$-KB $(\mathcal{T}, \mathcal{A})$ if $\mathcal{I}$ is a model for $\mathcal{T}$ and every assertion of $\mathcal{A}$ is satisfied in $\mathcal{I}$.

An (*essentially positive existential*) $\mathcal{L}$-*query* $q(x_1, \ldots, x_n)$ is a formula

$$\exists y_1 \cdots \exists y_m \varphi(x_1, \ldots, x_n, y_1, \ldots, y_m),$$

where φ is constructed, using only $\wedge$ and $\vee$, from atoms of the form $C(t)$ and $P(t_1, t_2)$, with C being an $\mathcal{L}$-concept, P a role, and t_i being either a variable from the list $x_1, \ldots, x_n, y_1, \ldots, y_m$ or an object name. Given an $\mathcal{L}$-KB $(\mathcal{T}, \mathcal{A})$ and an

$\mathcal{L}$-query $q(\boldsymbol{x})$, with $\boldsymbol{x} = x_1, \dots, x_n$, we say that an n-tuple $\boldsymbol{a}$ of object names is an *answer* to $q(\boldsymbol{x})$ w.r.t. $(\mathcal{T}, \mathcal{A})$ and write $(\mathcal{T}, \mathcal{A}) \models q(\boldsymbol{a})$ if, for every model $\mathcal{I}$ for $(\mathcal{T}, \mathcal{A})$, we have $\mathcal{I} \models q(\boldsymbol{a})$. The data complexity of the query answering problem for *DL-Lite*$_{horn}$ knowledge bases is in LogSpace, while for *DL-Lite*$_{bool}$ it is coNP-complete [8].

3 Types of Conservativity and Modularity

In this section, we introduce four different notions of conservative extension for *DL-Lite*$_{bool}$ and *DL-Lite*$_{horn}$, discuss their applications, investigate their meta-properties, and determine the computational complexity of the corresponding reasoning tasks. By a *signature* we understand here a finite set Σ of concept names and role names.[3] Given a concept, role, TBox, ABox, or query E, we denote by $\mathsf{sig}(E)$ the *signature* of E, that is, the set of concept and role names that occur in E. It is worth noting that the symbols $\bot$ and $\top$ are regarded as logical symbols. Thus, $\mathsf{sig}(\bot) = \mathsf{sig}(\top) = \emptyset$. A concept (role, TBox, ABox, query) E is called a *Σ-concept* (*role*, *TBox*, *ABox*, *query*, respectively) if $\mathsf{sig}(E) \subseteq \Sigma$. Thus, P^- is a Σ-role iff $P \in \Sigma$. As before, we will use $\mathcal{L}$ as a generic name for *DL-Lite*$_{bool}$ and *DL-Lite*$_{horn}$.

Definition 1 (deductive conservative extension). Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be $\mathcal{L}$-TBoxes and Σ a signature. We call $\mathcal{T}_1 \cup \mathcal{T}_2$ a (*deductive*) *conservative extension of* $\mathcal{T}_1$ *in* $\mathcal{L}$ *w.r.t.* Σ if, for every $\mathcal{L}$-concept inclusion $C_1 \sqsubseteq C_2$ with $\mathsf{sig}(C_1 \sqsubseteq C_2) \subseteq \Sigma$, we have $\mathcal{T}_1 \models C_1 \sqsubseteq C_2$ whenever $\mathcal{T}_1 \cup \mathcal{T}_2 \models C_1 \sqsubseteq C_2$.

This notion of deductive conservative extension is appropriate in the following situations; see also [2]. (i) Suppose that $\mathcal{T}_1$ is a TBox which does not cover part of its domain in sufficient detail. An ontology engineer, say Eve, decides to expand it by axioms $\mathcal{T}_2$, but wants to be sure that by doing this she does not interfere with the derivable inclusions between Σ-concepts. Then she should check whether $\mathcal{T}_1 \cup \mathcal{T}_2$ is a conservative extension of $\mathcal{T}_1$ in $\mathcal{L}$ w.r.t. to Σ. (ii) If the designer of an ontology $\mathcal{T}_2$ *imports* an ontology $\mathcal{T}_1$ and wants to ensure that no extra inclusions between $\mathsf{sig}(\mathcal{T}_1)$-concepts are derivable after importing the ontology, then again she should check whether $\mathcal{T}_1 \cup \mathcal{T}_2$ is a conservative extension of $\mathcal{T}_1$ in $\mathcal{L}$ w.r.t. $\mathsf{sig}(\mathcal{T}_1)$. Observe that in *DL-Lite*$_{bool}$, $\mathcal{T}_1 \cup \mathcal{T}_2$ is a deductive conservative extension of $\mathcal{T}_1$ iff every $\mathcal{T}_1$-satisfiable *DL-Lite*$_{bool}$ concept C with $\mathsf{sig}(C) \subseteq \Sigma$ is also $\mathcal{T}_1 \cup \mathcal{T}_2$-satisfiable.

Theorem 1. *For any DL-Lite*$_{horn}$ *TBoxes* $\mathcal{T}_1$, $\mathcal{T}_2$ *and any signature* Σ, *the following two conditions are equivalent:*

- $\mathcal{T}_1 \cup \mathcal{T}_2$ *is a conservative extension of* $\mathcal{T}_1$ *in DL-Lite*$_{bool}$ *w.r.t* Σ*;*
- $\mathcal{T}_1 \cup \mathcal{T}_2$ *is a conservative extension of* $\mathcal{T}_1$ *in DL-Lite*$_{horn}$ *w.r.t.* Σ*.*

[3] In the languages we consider, object names do not occur in TBoxes. Therefore, in this paper, we assume that signatures do not contain object names. When considering languages with nominals one would have to allow for object names in signatures.

For DL-Lite$_{horn}$ TBoxes, the problem of deciding whether $\mathcal{T}_1 \cup \mathcal{T}_2$ is a conservative extension of $\mathcal{T}_1$ in DL-Lite$_{horn}$ w.r.t Σ is coNP*-complete. For DL-Lite$_{bool}$ TBoxes, this problem is Π_2^p-complete.*

Observe that the complexity lower bounds follow immediately from the same lower bounds for the corresponding reasoning problems in classical propositional (Horn) logic. The upper bounds are proved in the Appendix. We remind the reader that conservativity is much harder for most DLs: it is ExpTime-complete for $\mathcal{EL}$ [9], 2ExpTime-complete for $\mathcal{ALC}$ and $\mathcal{ALCQI}$, and undecidable for $\mathcal{ALCQIO}$ [2, 4]. To explain, at a very high level, the reason for these results we consider the notion of a conservative extension in *DL-Lite$_{bool}$*: let $\mathcal{T}_1$ and $\mathcal{T}_2$ be TBoxes and Σ a signature with $\Sigma \subseteq \mathsf{sig}(\mathcal{T}_1)$. $\mathcal{T}_1 \cup \mathcal{T}_2$ is not a conservative extension of $\mathcal{T}_1$ in *DL-Lite$_{bool}$* w.r.t. Σ if, and only if, there exists a concept C with $\mathsf{sig}(C) \subseteq \Sigma$ such that C is satisfiable relative to $\mathcal{T}_1$ but not relative to $\mathcal{T}_1 \cup \mathcal{T}_2$. We call such a concept C a *witness-concept*. Thus, a decision procedure for conservativity can be regarded as a systematic search for such a witness-concept. In standard description logics such as *DL-Lite$_{bool}$*, $\mathcal{EL}$, $\mathcal{ALC}$, etc. the space of all possible witnesses is infinite. (This observation implies that from the decidability of the problem whether a concept is satisfiable w.r.t. a TBox it does not necessarily follow that conservativity is decidable.) Now, we prove in the Appendix that for *DL-Lite$_{bool}$* the existence of some witness concept implies the existence of a witness concept of size *polynomial* in the size of $\mathcal{T}_1$ and $\mathcal{T}_2$ and which uses only the numeral parameters which occur in number restrictions from $\mathcal{T}_1 \cup \mathcal{T}_2$. In contrast, in $\mathcal{EL}$ one can construct examples in which minimal witnesses for non-conservativity are of double exponential size in the size of $\mathcal{T}_1$ and $\mathcal{T}_2$ [9]. In $\mathcal{ALC}$, one can even enforce minimal witness concepts of triple exponential size [2]. The reason for this difference is the availability of *qualified* quantification in those language, and its absence in *DL-Lite$_{bool}$*. The result on the size of witness concepts for *DL-Lite$_{bool}$* is easily converted into a decision procedure for non-conservativity which is in Π_2^p: just (non-deterministically) guess a concept C of polynomial size in the size of $\mathcal{T}_1$ and $\mathcal{T}_2$ and with $\mathsf{sig}(C) \subseteq \Sigma$ and check, by calling an NP-oracle, whether (i) C is satisfiable w.r.t. $\mathcal{T}_1$ and (ii) not satisfiable w.r.t. $\mathcal{T}_1 \cup \mathcal{T}_2$. Because of the larger size of minimal witnesses, no such procedure exists for $\mathcal{EL}$ or $\mathcal{ALC}$.

Consider now the situation when the ontology designer is not only interested in preserving derivable concept inclusions, but also in preserving answers to queries, for both *DL-Lite$_{bool}$* and *DL-Lite$_{horn}$* TBoxes.

Definition 2 (query conservative extension). Let $\mathcal{T}_1$, $\mathcal{T}_2$ be $\mathcal{L}$-TBoxes and Σ a signature. We call $\mathcal{T}_1 \cup \mathcal{T}_2$ a *query conservative extension of* $\mathcal{T}_1$ *in* $\mathcal{L}$ *w.r.t.* Σ if, for every $\mathcal{L}$-ABox $\mathcal{A}$ with $\mathsf{sig}(\mathcal{A}) \subseteq \Sigma$, every $\mathcal{L}$-query q with $\mathsf{sig}(q) \subseteq \Sigma$, and every tuple $\boldsymbol{a}$ of object names from $\mathcal{A}$, we have $(\mathcal{T}_1, \mathcal{A}) \models q(\boldsymbol{a})$ whenever $(\mathcal{T}_1 \cup \mathcal{T}_2, \mathcal{A}) \models q(\boldsymbol{a})$.

It is easy to see that query conservativity implies deductive conservativity for both logics *DL-Lite$_{bool}$* and *DL-Lite$_{horn}$*. Indeed, let $\mathcal{L}$ be one of *DL-Lite$_{bool}$* and *DL-Lite$_{horn}$*. Suppose that we have $\mathcal{T}_1 \not\models C_1 \sqsubseteq C_2$ but $\mathcal{T}_1 \cup \mathcal{T}_2 \models C_1 \sqsubseteq C_2$, for some $\mathcal{L}$-concept inclusion $C_1 \sqsubseteq C_2$ with $\mathsf{sig}(C_1 \sqsubseteq C_2) \subseteq \Sigma$. Consider the ABox $\mathcal{A} = \{C_1(a)\}$ and the query $q = C_2(a)$. Then clearly $(\mathcal{T}_1 \cup \mathcal{T}_2, \mathcal{A}) \models q$, while $(\mathcal{T}_1, \mathcal{A}) \not\models q$. Note that in *DL-Lite*$_{horn}$, $C_1 = B_1 \sqcap \dots \sqcap B_k$ and $C_2 = B$, where $B, B_1, \dots, B_k$ are basic concepts.

The following example shows, in particular, that the converse implication does not hold.

Example 1. (1) To see that there are deductive conservative extensions which are not query conservative, take $\mathcal{T}_1 = \emptyset$, $\mathcal{T}_2 = \{A \sqsubseteq \exists P, \exists P^- \sqsubseteq B\}$ and $\Sigma = \{A, B\}$. Then $\mathcal{T}_2$ is a deductive conservative extension of $\mathcal{T}_1$ (in both *DL-Lite*$_{bool}$ and *DL-Lite*$_{horn}$) w.r.t. Σ. However, it is not a query conservative extension: let $\mathcal{A} = \{A(a)\}$ and $q = \exists y\, B(y)$; then $(\mathcal{T}_1, \mathcal{A}) \not\models q$ but $(\mathcal{T}_2, \mathcal{A}) \models q$.

(2) Note also that query conservativity in *DL-Lite*$_{horn}$ does not imply query conservativity in *DL-Lite*$_{bool}$. Indeed, let $\mathcal{T}_1 = \emptyset$, $\mathcal{T}_2 = \{A \sqsubseteq \exists P, A \sqcap \exists P^- \sqsubseteq \bot\}$ and $\Sigma = \{A\}$. Then $\mathcal{T}_2$ is not a query conservative extension of $\mathcal{T}_1$ in *DL-Lite*$_{bool}$ w.r.t. Σ: just take $\mathcal{A}$ as before and $q = \exists y \, \neg A(y)$. But it is a query conservative extension in *DL-Lite*$_{horn}$.

In the definition of essentially positive existential queries for *DL-Lite*$_{bool}$ above, we have allowed negated concepts in queries and ABoxes. An alternative approach would be to allow only *positive* concepts. These two types of queries give rise to different notions of query conservativity: under the second definition, the TBox $\mathcal{T}_2$ from Example 1 (2) is a query conservative extension of $\mathcal{T}_1 = \emptyset$ w.r.t. $\{A\}$, even in *DL-Lite*$_{bool}$. We argue, however, that it is the *essentially positive* queries that should be considered in the context of this investigation. The reason is that, with positive queries, the addition of the *definition* $B \equiv \neg A$ to $\mathcal{T}_2$ and B to Σ would result in a TBox which is not a query conservative extension in *DL-Lite*$_{bool}$ of $\mathcal{T}_1$ any longer. This kind of non-robust behaviour of the notion of conservativity is clearly undesirable. Obviously, the definitions we gave are robust under the addition of such definitions. Moreover, two extra robustness conditions hold true.

Definition 3 (robustness). Let Σ be a signature and cons_Σ a 'conservativity' relation between TBoxes w.r.t. Σ. (For example, $\mathsf{cons}_\Sigma(\mathcal{T}_1, \mathcal{T}_1 \cup \mathcal{T}_2)$ can be defined as '$\mathcal{T}_1 \cup \mathcal{T}_2$ is a deductive conservative extension of $\mathcal{T}_1$ in *DL-Lite*$_{bool}$ w.r.t. Σ').

- We say that cons_Σ is *robust under joins* if $(\mathcal{T}_0, \mathcal{T}_0 \cup \mathcal{T}_1), (\mathcal{T}_0, \mathcal{T}_0 \cup \mathcal{T}_2) \in \mathsf{cons}_\Sigma$ and $\mathsf{sig}(\mathcal{T}_1) \cap \mathsf{sig}(\mathcal{T}_2) \subseteq \Sigma$ imply $(\mathcal{T}_0, \mathcal{T}_0 \cup \mathcal{T}_1 \cup \mathcal{T}_2) \in \mathsf{cons}_\Sigma$.
- We say that cons_Σ is *robust under vocabulary extensions* if $(\mathcal{T}_1, \mathcal{T}_1 \cup \mathcal{T}_2) \in \mathsf{cons}_\Sigma$ implies $(\mathcal{T}_1, \mathcal{T}_1 \cup \mathcal{T}_2) \in \mathsf{cons}_{\Sigma'}$, for all signatures Σ' with $\mathsf{sig}(\mathcal{T}_1 \cup \mathcal{T}_2) \cap \Sigma' \subseteq \Sigma$.

Roughly speaking, robustness under joins means that an ontology can be safely imported into joins of independent ontologies if each of them safely imports the ontology: if the shared symbols of $\mathcal{T}_1$ and $\mathcal{T}_2$ are from Σ, and both $\mathcal{T}_1 \cup \mathcal{T}_0$ and $\mathcal{T}_2 \cup \mathcal{T}_0$ are conservative extensions of $\mathcal{T}_0$ w.r.t. Σ, then the join $\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_0$ is a conservative extension of $\mathcal{T}_0$ w.r.t. Σ. In practice, this property supports collaborative ontology development in the following sense: it implies that if two (or more) ontology developers extend a given ontology $\mathcal{T}_0$ independently and do not use common symbols with the exception of those in a certain signature Σ then they can safely form the union of $\mathcal{T}_0$ and all their additional axioms provided that their individual extensions are safe for Σ (in the sense of deductive or, respectively, query conservativity). This property is closely related to the well-known *Robinson consistency lemma* and *interpolation* (see e.g., [10]) and has been investigated in the context of modular software specification [11] as well. We refer the reader to the Appendix for a more detailed discussion.

Robustness under vocabulary extensions is even closer to interpolation: it states that once we know conservativity w.r.t. Σ, we also know conservativity with respect to any signature with extra fresh symbols. The practical relevance of this property is as follows: when specifying the signature Σ for which an ontology developer wants to check conservativity, the developer only has to decide which symbols from $\mathcal{T}_1$ and $\mathcal{T}_2$ she wants to include into Σ. The answer to the query does not depend on whether Σ contains fresh symbols or not.

Theorem 2. *Both deductive and query conservativity in both DL-Lite$_{bool}$ and DL-Lite$_{horn}$ are robust under joins and vocabulary extensions.*

Actually, in *DL-Lite$_{horn}$* and *DL-Lite$_{bool}$* we even have a much stronger form of interpolation which is known as the *uniform interpolation property* [12, 13]. Let $\mathcal{T}$ be a TBox and Σ a signature. A TBox $\mathcal{T}'$ is called a *uniform interpolant* for $\mathcal{T}$ w.r.t. Σ in $\mathcal{L}$ if $\mathcal{T}'$ is an $\mathcal{L}$-TBox, $\mathsf{sig}(\mathcal{T}') \subseteq \Sigma$, $\mathcal{T} \models \mathcal{T}'$, and for all $\mathcal{L}$-concept inclusions $C_1 \sqsubseteq C_2$ with $\mathcal{T} \models C_1 \sqsubseteq C_2$ and $\mathsf{sig}(C_1, C_2) \cap \mathsf{sig}(\mathcal{T}) \subseteq \Sigma$, we have $\mathcal{T}' \models C_1 \sqsubseteq C_2$.

Intuitively, a uniform interpolant for $\mathcal{T}$ w.r.t. Σ contains exactly the same information about Σ in terms of concept inclusions as $\mathcal{T}$ *without using additional* symbols. For most DLs, such as $\mathcal{ALC}$, uniform interpolants do not necessarily exist [2].

Theorem 3. *Let $\mathcal{L}$ be DL-Lite$_{horn}$ or DL-Lite$_{bool}$. Then for every $\mathcal{L}$-TBox $\mathcal{T}$ and every signature Σ there exists a uniform interpolant for $\mathcal{T}$ w.r.t. Σ in $\mathcal{L}$.*

We note that one has to be very careful when interpreting the meaning of uniform interpolants. Consider, for instance, $\mathcal{T} = \{A \sqsubseteq \exists P,\ A \sqcap \exists P^- \sqsubseteq \bot\}$ and $\Sigma = \{A\}$. The TBox $\mathcal{T}' = \emptyset$ is a uniform interpolant of $\mathcal{T}$ w.r.t. Σ in *DL-Lite$_{bool}$*. However, as we saw in Example 1, the TBoxes $\mathcal{T}$ and $\mathcal{T}'$ behave differently with respect to queries in Σ: $(\mathcal{T}, \{A(a)\}) \models \exists x\, \neg A(x)$ but $(\mathcal{T}', \{A(a)\}) \not\models \exists x\, \neg A(x)$.

As sketched above, one application of deductive and query conservativity is to ensure that, when importing an ontology $\mathcal{T}$, one does not change the meaning of its vocabulary (in terms of concept inclusions or answers to queries). We now consider the situation where the ontology $\mathcal{T}$ to be imported is not known because, for example, it is still under development or because different ontologies can be chosen. In this case, $\mathcal{T}$ should be regarded as a 'black box' which supplies information about a signature Σ. To ensure that the meaning of the symbols in

Σ as defined by this black box is not changed by importing it into $\mathcal{T}_1$, one has to check the following condition:

Definition 4 (safety). Let Σ be a signature and $\mathcal{T}_1$ an $\mathcal{L}$-TBox. We say that $\mathcal{T}_1$ is *safe for Σ w.r.t. deductive (or query) conservativity in $\mathcal{L}$* if, for all $\mathcal{L}$-TBoxes $\mathcal{T}$ with $\mathsf{sig}(\mathcal{T}) \cap \mathsf{sig}(\mathcal{T}_1) \subseteq \Sigma$, $\mathcal{T}_1 \cup \mathcal{T}$ is a deductive (respectively, query) conservative extension of $\mathcal{T}$ in $\mathcal{L}$ w.r.t. Σ.

This notion has been introduced in [3] where the reader can find further discussion. A natural generalisation of safety, considered in [5], is the following property:

Definition 5 (strong deductive/query conservative extension). Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be $\mathcal{L}$-TBoxes and Σ a signature. We call $\mathcal{T}_1 \cup \mathcal{T}_2$ a *strong deductive (query) conservative extension* of $\mathcal{T}_1$ in $\mathcal{L}$ w.r.t. Σ if $\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}$ is a deductive (respectively, query) conservative extension of $\mathcal{T}_1 \cup \mathcal{T}$ in $\mathcal{L}$ w.r.t. Σ, for every $\mathcal{L}$-TBox $\mathcal{T}$ with $\mathsf{sig}(\mathcal{T}) \cap \mathsf{sig}(\mathcal{T}_1 \cup \mathcal{T}_2) \subseteq \Sigma$.

Observe that safety is indeed a special case of strong conservativity: it covers exactly the case where the TBox $\mathcal{T}_1$ in the definition of strong conservativity is empty. A typical application of strong conservativity for ontology re-use is as follows (see [5]). Suppose that there is a large ontology $\mathcal{O}$ and a subset Σ of its signature. Assume also that the ontology designer wants to use what $\mathcal{O}$ says about Σ in her own ontology $\mathcal{T}$ she is developing at the moment. Then instead of importing $\mathcal{O}$ as a whole, it would be preferable to find a small subset $\mathcal{T}_1$ of $\mathcal{O}$, which says precisely the same about Σ as $\mathcal{O}$ does, and import only this small $\mathcal{T}_1$ rather than the large $\mathcal{O}$. But then what are the conditions we should impose on $\mathcal{T}_1$? An obvious minimal requirement is that by importing $\mathcal{T}_1$ into $\mathcal{T}$ we obtain the same consequences for subsumptions/queries over Σ as by importing $\mathcal{O}$ into $\mathcal{T}$. Depending on whether concept inclusions or answers to queries are of interest, one therefore wants $\mathcal{O} = \mathcal{T}_1 \cup \mathcal{T}_2$ to be a strong deductive or query conservative extension of $\mathcal{T}_1$ w.r.t Σ. We refer the reader to [5] for further discussion and algorithms for extracting such TBoxes from a given TBox.

Example 2. (1) Let us see first that strong deductive conservativity is indeed a stronger notion than deductive conservativity, for *DL-Lite*$_{bool}$ and *DL-Lite*$_{horn}$. Consider again the TBoxes $\mathcal{T}_1 = \emptyset$, $\mathcal{T}_2 = \{A \sqsubseteq \exists R, \quad A \sqcap \exists R^- \sqsubseteq \bot\}$, and $\Sigma = \{A\}$. Then $\mathcal{T}_1 \cup \mathcal{T}_2$ is a deductive conservative extension of $\mathcal{T}_1$ w.r.t. Σ. However, $\mathcal{T}_1 \cup \mathcal{T}_2$ is *not* a strong deductive conservative extension of $\mathcal{T}_1$ w.r.t. Σ. Let $\mathcal{T} = \{\top \sqsubseteq A\}$. Then we have $\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T} \models \top \sqsubseteq \bot$ but $\mathcal{T}_1 \cup \mathcal{T} \not\models \top \sqsubseteq \bot$.

(2) We show now that an analogue of Theorem 1 does not hold for strong deductive conservativity. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be the following *DL-Lite*$_{horn}$ TBoxes

$$\mathcal{T}_1 = \big\{A \sqcap B \sqsubseteq \bot, \quad \top \sqsubseteq \exists P_1, \quad \top \sqsubseteq \exists P_2, \quad \exists P_1^- \sqsubseteq A, \quad \exists P_2^- \sqsubseteq B\big\},$$
$$\mathcal{T}_2 = \big\{\top \sqsubseteq \exists R, \quad A \sqcap \exists R^- \sqsubseteq \bot, \quad B \sqcap \exists R^- \sqsubseteq \bot\big\},$$

and let $\Sigma = \{A, B, P_1, P_2\}$. $\mathcal{T}_2$ says that $\top \not\sqsubseteq A \sqcup B$. Now, in *DL-Lite*$_{bool}$, $\mathcal{T}_1 \cup \mathcal{T}_2$ is not a strong deductive conservative extension of $\mathcal{T}_1$ w.r.t. Σ: just take

$\mathcal{T} = \{\top \sqsubseteq A \sqcup B\}$. However, $\mathcal{T}_1 \cup \mathcal{T}_2$ is a strong deductive conservative extension of $\mathcal{T}_1$ in *DL-Lite$_{horn}$*.

Obviously, the robustness conditions introduced above are of importance for the strong versions of conservativity as well.

Theorem 4. *Both strong deductive and strong query conservativity are robust under joins and vocabulary extensions for DL-Lite$_{bool}$ and DL-Lite$_{horn}$.*

In addition to these types of robustness, the following condition, which is dual to robustness under joins, is of crucial importance for iterated applications of the notion of safety for a signature. Suppose that $\mathcal{T}$ is safe for $\Sigma_1 \cup \Sigma_2$ under some notion of conservativity and $\Sigma_1 \cap \Sigma_2 = \emptyset$. Then, for any $\mathcal{T}_1$ with $\mathsf{sig}(\mathcal{T}_1) \cap (\mathsf{sig}(\mathcal{T}) \cup \Sigma_2) \subseteq \Sigma_1$, the TBox $\mathcal{T} \cup \mathcal{T}_1$ should be safe for Σ_2 for the same notion of conservativity. Without this property, one might have the situation that a TBox is safe for a signature $\Sigma_1 \cup \Sigma_2$, but after importing a TBox for Σ_1 the resulting TBox is not safe for Σ_2 any longer, which is clearly undesirable.

Theorem 5 (robustness under joins of signatures). *Let $\mathcal{L}$ be either DL-Lite$_{bool}$ or DL-Lite$_{horn}$. If an $\mathcal{L}$-TBox $\mathcal{T}$ is safe for a signature $\Sigma_1 \cup \Sigma_2$ w.r.t. deductive/query conservativity in $\mathcal{L}$, $\Sigma_1 \cap \Sigma_2 = \emptyset$ and $\mathcal{T}_1$ is a satisfiable $\mathcal{L}$-TBox with* $\mathsf{sig}(\mathcal{T}_1) \cap (\mathsf{sig}(\mathcal{T}) \cup \Sigma_2) \subseteq \Sigma_1$, *then $\mathcal{T} \cup \mathcal{T}_1$ is safe for Σ_2 w.r.t. deductive/query-conservativity in $\mathcal{L}$.*

This result follows immediately from the fact that any two satisfiable $\mathcal{L}$-TBoxes in disjoint signatures are strong query conservative extensions of each other. This property fails for a number of stronger notions of conservativity, for example, model conservativity.

The next theorem shows that in all those cases where we have not provided counterexamples the introduced notions of conservativity are equivalent:

DL-Lite$_{horn}$	deductive	$\lneqq$	query	$\lneqq$	strong deductive	$\equiv$	strong query
DL-Lite$_{bool}$	deductive	$\lneqq$	query	$\equiv$	strong deductive	$\equiv$	strong query

It also establishes the complexity of the corresponding decision problems.

Theorem 6. *Let $\mathcal{L}$ be either DL-Lite$_{bool}$ or DL-Lite$_{horn}$, $\mathcal{T}_1$ and $\mathcal{T}_2$ $\mathcal{L}$-TBoxes, and Σ a signature.*

For $\mathcal{L}$ = DL-Lite$_{bool}$, the following conditions are equivalent:

(1) *$\mathcal{T}_1 \cup \mathcal{T}_2$ is a query conservative extension of $\mathcal{T}_1$ in $\mathcal{L}$ w.r.t. Σ;*
(2) *$\mathcal{T}_1 \cup \mathcal{T}_2$ is a strong deductive conservative extension of $\mathcal{T}_1$ in $\mathcal{L}$ w.r.t. Σ;*
(3) *$\mathcal{T}_1 \cup \mathcal{T}_2$ is a strong query conservative extension of $\mathcal{T}_1$ in $\mathcal{L}$ w.r.t. Σ.*

For $\mathcal{L}$ = DL-Lite$_{horn}$, conditions (2) *and* (3) *are equivalent, while* (1) *is strictly weaker than each of them.*

For DL-Lite$_{horn}$, the decision problems corresponding to conditions (1)–(3) *are all* coNP*-complete. For DL-Lite$_{bool}$, these problems are Π^p_2-complete.*

The lower bounds again follow immediately from the corresponding lower bounds for propositional logic. The upper bounds and equivalences are proved in the Appendix. The proofs are based on the observation from [8] that reasoning in *DL-Lite*$_{bool}$ and *DL-Lite*$_{horn}$ can be reduced to reasoning in propositional logic and reasoning about the existence of outgoing and incoming arrows for domain elements in interpretations. We believe that the equivalences stated in Theorem 6 are somewhat surprising. For example, it can be easily seen that for $\mathcal{ALC}$ none of those equivalences holds true.

4 Model-Theoretic Characterisations of Conservativity

All the results discussed above are proved with the help of the model-theoretic characterisations of our notions of conservativity formulated below.

For a TBox $\mathcal{T}$, denote by $Q_{\mathcal{T}}$ the set of all numerical parameters occurring in $\mathcal{T}$ together with 1.

Let Σ be a signature and Q a set of positive natural numbers containing 1. By a *ΣQ-concept* we mean any concept of the form $\bot$, $\top$, A_i, $\geq q\, R$, or its negation for some $A_i \in \Sigma$, Σ-role R and $q \in Q$. A *ΣQ-type* is a set $\boldsymbol{t}$ of ΣQ-concepts containing $\top$ such that the following conditions hold:

- for every ΣQ-concept C, either $C \in \boldsymbol{t}$ or $\neg C \in \boldsymbol{t}$,
- if $q < q'$ are both in Q and $\geq q'\, R \in \boldsymbol{t}$ then $\geq q\, R \in \boldsymbol{t}$,
- if $q < q'$ are both in Q and $\neg(\geq q\, R) \in \boldsymbol{t}$ then $\neg(\geq q'\, R) \in \boldsymbol{t}$.

It should be clear that, for each ΣQ-type $\boldsymbol{t}$ with $\bot \notin \boldsymbol{t}$, there is an interpretation $\mathcal{I}$ and a point x in it such that, for every $C \in \boldsymbol{t}$, we have $x \in C^{\mathcal{I}}$. In this case we say that $\boldsymbol{t}$ is *realised at x in $\mathcal{I}$*, or that $\boldsymbol{t}$ is the *ΣQ-type of x in $\mathcal{I}$*.

Definition 6. A set Ξ of ΣQ-types is said to be *$\mathcal{T}$-realisable* if there is a model for $\mathcal{T}$ realising *all* types from Ξ. We also say that Ξ is *precisely $\mathcal{T}$-realisable* if there is a model $\mathcal{I}$ for $\mathcal{T}$ such that $\mathcal{I}$ realises all types in Ξ and every ΣQ-type realised in $\mathcal{I}$ is in Ξ.

Given a signature Σ, we say that interpretations $\mathcal{I}$ and $\mathcal{J}$ are *Σ-isomorphic* and write $\mathcal{I} \sim_{\Sigma} \mathcal{J}$ if there is a bijection $f\colon \Delta^{\mathcal{I}} \to \Delta^{\mathcal{J}}$ such that $f(a^{\mathcal{I}}) = a^{\mathcal{J}}$, for every object name a, $x \in A^{\mathcal{I}}$ iff $f(x) \in A^{\mathcal{J}}$, for every concept name $A \in \Sigma$, and $(x, y) \in P^{\mathcal{I}}$ iff $(f(x), f(y)) \in P^{\mathcal{J}}$, for every role name $P \in \Sigma$. Clearly, Σ-isomorphic interpretations cannot be distinguished by TBoxes, ABoxes, or queries *over Σ*.

Given a set $\mathcal{I}_i$, $i \in I$, of interpretations with $1 \in I$, define the interpretation (the *disjoint union* of the $\mathcal{I}_i$) $\mathcal{J} = \bigoplus_{i \in I} \mathcal{I}_i$, where $\Delta^{\mathcal{J}} = \{(i, w) \mid i \in I, w \in \Delta_i\}$, $a^{\mathcal{J}} = (1, a^{\mathcal{I}_1})$, for every object name a, $A^{\mathcal{J}} = \{(i, w) \mid w \in A^{\mathcal{I}_i}\}$, for every concept name A, and $P^{\mathcal{J}} = \{((i, w_1), (i, w_2)) \mid (w_1, w_2) \in P^{\mathcal{I}_i}\}$, for every role name P. Given an interpretation $\mathcal{I}$, we set $\mathcal{I}^{\omega} = \bigoplus_{i \in \omega} \mathcal{I}_i$, where $\mathcal{I}_i = \mathcal{I}$ for $i \in \omega$. Again, it should be clear that TBoxes, ABoxes or queries (over any signature) cannot distinguish between $\mathcal{I}$ and $\mathcal{I}^{\omega}$.

The following lemma provides an important model-theoretic property of *DL-Lite*$_{bool}$ which is used to establish model-theoretic characterisations of various notions of conservativity.

Lemma 1. *Let $\mathcal{J}$ be an (at most countable) model for $\mathcal{T}_1$ and Σ a signature with $\Sigma \subseteq \mathsf{sig}(\mathcal{T}_1)$. Suppose that there is a model for $\mathcal{T}_1 \cup \mathcal{T}_2$ realising precisely the same $\Sigma Q_{\mathcal{T}_1 \cup \mathcal{T}_2}$-types as $\mathcal{J}$. Then there is a model $\mathcal{I}^*$ for $\mathcal{T}_1 \cup \mathcal{T}_2$ such that $\mathcal{I}^* \sim_\Sigma \mathcal{J}^\omega$.*

In particular, $\mathcal{I}^* \models \mathcal{A}$ iff $\mathcal{J} \models \mathcal{A}$, for all ABoxes $\mathcal{A}$ over Σ, $\mathcal{I}^* \models \mathcal{T}$ iff $\mathcal{J} \models \mathcal{T}$, for all TBoxes $\mathcal{T}$ over Σ, and $\mathcal{I}^* \models q(\boldsymbol{a})$ iff $\mathcal{J} \models q(\boldsymbol{a})$, for all queries $q(\boldsymbol{a})$ over Σ.

In the case of *DL-Lite*$_{horn}$ we need some extra definitions. By a *ΣQ^h-concept* we mean any concept of the form $\bot$, A_i or $\geq q\, R$, for some $A_i \in \Sigma$, Σ-role R and $q \in Q$. Given a ΣQ-type $\boldsymbol{t}$, let

$$\boldsymbol{t}^+ = \{B \in \boldsymbol{t} \mid B \text{ a } \Sigma Q^h\text{-concept}\}.$$

Say that a ΣQ-type $\boldsymbol{t}_1$ is *h-contained* in a ΣQ-type $\boldsymbol{t}_2$ if $\boldsymbol{t}_1^+ \subseteq \boldsymbol{t}_2^+$. The following two notions characterise conservativity for *DL-Lite*$_{horn}$:

Definition 7. A set Ξ of ΣQ-types is said to be *sub-precisely $\mathcal{T}$-realisable* if there is a model $\mathcal{I}$ for $\mathcal{T}$ such that $\mathcal{I}$ realises all types from Ξ, and every ΣQ-type realised in $\mathcal{I}$ is h-contained in some type from Ξ. We also say that Ξ is *join-precisely $\mathcal{T}$-realisable* if there is a model $\mathcal{I}$ for $\mathcal{T}$ such that, for every ΣQ-type $\boldsymbol{t}$ realised in $\mathcal{I}$, $\Xi_{\boldsymbol{t}} = \{\boldsymbol{t}_i \in \Xi \mid \boldsymbol{t}^+ \subseteq \boldsymbol{t}_i^+\} \neq \emptyset$ and

$$\boldsymbol{t}^+ = \bigcap_{\boldsymbol{t}_i \in \Xi_{\boldsymbol{t}}} \boldsymbol{t}_i^+.$$

(It follows, in particular, that $\boldsymbol{t}^+ \subseteq \boldsymbol{t}_i^+$, for each $\boldsymbol{t}_i \in \Xi_{\boldsymbol{t}}$, and therefore, Ξ is sub-precisely $\mathcal{T}$-realisable.) As it is shown in the Appendix, once can deterministically check in polynomial time whether a given Ξ is join-precisely $\mathcal{T}$-realisable.

The table below gives characterisations of our four notions of conservativity in the following form: $\mathcal{T}_1 \cup \mathcal{T}_2$ is an α conservative extension of $\mathcal{T}_1$ w.r.t. Σ in $\mathcal{L}$ if, and only if, every precisely $\mathcal{T}_1$-realisable set Ξ of $\Sigma Q_{\mathcal{T}_1 \cup \mathcal{T}_2}$ types is '. . . '

5 Conclusion

We have analysed the relation between different notions of conservative extension in description logics *DL-Lite*$_{bool}$ and *DL-Lite*$_{horn}$, and proved that the corresponding reasoning problems are not harder than the same problems in propositional logic. Moreover, we have also shown that important meta-properties for modular ontology engineering, such as robustness under joins, vocabulary extensions, and iterated import of ontologies, hold true for these notions of conservativity.

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conservativity α	language $\mathcal{L}$	
	DL-Lite$_{bool}$	*DL-Lite*$_{horn}$
deductive	$\mathcal{T}_1 \cup \mathcal{T}_2$-realisable	$\mathcal{T}_1 \cup \mathcal{T}_2$-realisable
query	precisely $\mathcal{T}_1 \cup \mathcal{T}_2$-realisable	sub-precisely $\mathcal{T}_1 \cup \mathcal{T}_2$-realisable
strong deductive		join-precisely $\mathcal{T}_1 \cup \mathcal{T}_2$-realisable
strong query		

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A Model-theoretic properties of *DL-Lite*

In this section, we introduce an 'amalgamation'-result for DL-Lite models. As a first application, we prove Theorem 3. The construction will be used throughout the paper.

Let Σ be a signature and Q a set of positive natural numbers containing 1. By a *ΣQ-concept* we mean any concept of the form $\bot$, $\top$, A_i, $\geq q\, R$, or its negation for some $A_i \in \Sigma$, Σ-role R and $q \in Q$. A *ΣQ-type* is a set $\boldsymbol{t}$ of ΣQ-concepts containing $\top$ such that the following conditions hold:

- for every ΣQ-concept C, either $C \in \boldsymbol{t}$ or $\neg C \in \boldsymbol{t}$,
- if $q < q'$ are both in Q and $\geq q'\, R \in \boldsymbol{t}$ then $\geq q\, R \in \boldsymbol{t}$,
- if $q < q'$ are both in Q and $\neg(\geq q\, R) \in \boldsymbol{t}$ then $\neg(\geq q'\, R) \in \boldsymbol{t}$.

It should be clear that, for each ΣQ-type $\boldsymbol{t}$ with $\bot \notin \boldsymbol{t}$, there is an interpretation $\mathcal{I}$ and a point x in it such that, for every $C \in \boldsymbol{t}$, we have $x \in C^{\mathcal{I}}$. In this case we say that $\boldsymbol{t}$ is *realised at x in $\mathcal{I}$*, or that $\boldsymbol{t}$ is the *ΣQ-type of x in $\mathcal{I}$* and denote it by $\boldsymbol{t}_{\mathcal{I}}^{\Sigma Q}(x)$. Given a TBox $\mathcal{T}$, a ΣQ-type $\boldsymbol{t}$ is called *$\mathcal{T}$-consistent* if $\boldsymbol{t}$ is realised in a model for $\mathcal{T}$. A set Ξ of ΣQ-types is said to be *$\mathcal{T}$-realisable* if there is a model for $\mathcal{T}$ realising *all* types from Ξ.

We say that a set Ξ of ΣQ-types is *precisely $\mathcal{T}$-realisable* if there is a model $\mathcal{I}$ for $\mathcal{T}$ such that $\mathcal{I}$ realises all types in Ξ, and every ΣQ-type realised in $\mathcal{I}$ is in Ξ. In this case we say that $\mathcal{I}$ *realises precisely* the types in Ξ.

Our next aim is to introduce, in Lemma 2, an operation which allows us to amalgamate interpretations in a 'truth-preserving' way. To do this we require two simple definitions.

Given a signature Σ, we say that two interpretations $\mathcal{I}$ and $\mathcal{J}$ are *Σ-isomorphic* and write $\mathcal{I} \sim_\Sigma \mathcal{J}$ if there is a bijection $f\colon \Delta^{\mathcal{I}} \to \Delta^{\mathcal{J}}$ such that $f(a^{\mathcal{I}}) = a^{\mathcal{J}}$, for every object name a, $x \in A^{\mathcal{I}}$ iff $f(x) \in A^{\mathcal{J}}$, for every concept name A in Σ, and $(x, y) \in P^{\mathcal{I}}$ iff $(f(x), f(y)) \in P^{\mathcal{J}}$, for every role name P in Σ. Clearly, Σ-isomorphic interpretations cannot be distinguished by TBoxes, ABoxes, or queries *over* Σ.

We will use the disjoint union construction: given a set $\mathcal{I}_i$, $i \in I$, of interpretations and $1 \in I$ define the interpretation

$$\mathcal{J} = \bigoplus_{i \in I} \mathcal{I}_i,$$

where $\Delta^{\mathcal{J}} = \{(i, w) \mid i \in I, w \in \Delta_i\}$, $a^{\mathcal{J}} = (1, a^{\mathcal{I}_1})$, for all object names a, $A^{\mathcal{J}} = \{(i, w) \mid w \in A^{\mathcal{I}_i}\}$, for all concept names A, and $P^{\mathcal{J}} = \{((i, w_1), (i, w_2)) \mid (w_1, w_2) \in P^{\mathcal{I}_i}\}$, for all role names P. Given an interpretation $\mathcal{I}$, we set

$$\mathcal{I}^\omega = \bigoplus_{i \in \omega} \mathcal{I}_i$$

where $\mathcal{I}_i = \mathcal{I}$ for $i \in \omega$. Again, it should be clear that TBoxes, ABoxes or queries (over any signature) cannot distinguish between $\mathcal{I}$ and $\mathcal{I}^\omega$.

For a TBox $\mathcal{T}$, denote by $Q_{\mathcal{T}}$ the set of all numerical parameters occurring in $\mathcal{T}$ together with 1.

The following lemma provides an important model-theoretic property of *DL-Lite*$_{bool}$ that will be frequently used to establish model-theoretic characterisations of various notions of conservativity.

Lemma 2. *Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be (at most countable) models for TBoxes $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively, and let Σ be a signature such that $\mathsf{sig}(\mathcal{T}_1) \cap \mathsf{sig}(\mathcal{T}_2) \subseteq \Sigma$. If interpretations $\mathcal{I}_1$ and $\mathcal{I}_2$ realise precisely the same $\Sigma Q_{\mathcal{T}_1 \cup \mathcal{T}_2}$-types, then there is an interpretation $\mathcal{I}^*$ such that:*

- $\mathcal{I}^* \models \mathcal{T}_1 \cup \mathcal{T}_2$,
- $\mathcal{I}^* \sim_\Sigma \mathcal{I}_1^\omega$, *and*
- $\mathcal{I}^*$, $\mathcal{I}_1$ *and* $\mathcal{I}_2$ *realise the same set of* $\Sigma Q_{\mathcal{T}_1 \cup \mathcal{T}_2}$*-types.*

Proof. Let Ξ be the set of $\Sigma Q_{\mathcal{T}_1 \cup \mathcal{T}_2}$-types realised in $\mathcal{I}_1$ (and $\mathcal{I}_2$). We show that $\mathcal{I}_1^\omega$ can be expanded to a model $\mathcal{I}^*$ for $\mathcal{T}_1 \cup \mathcal{T}_2$. As both $\mathcal{I}_1^\omega$ and $\mathcal{I}_2^\omega$ realise each $\Sigma Q_{\mathcal{T}_1 \cup \mathcal{T}_2}$-type from Ξ by countably infinitely many points, there is a bijection $f\colon \Delta^{\mathcal{I}_2^\omega} \to \Delta^{\mathcal{I}_1^\omega}$ which is *invariant under* $\Sigma Q_{\mathcal{T}_1 \cup \mathcal{T}_2}$-types. Now, we set $\Delta^{\mathcal{I}^*} = \Delta^{\mathcal{I}_1^\omega}$ and, for all object names a, concept names A, and role names P,

$$a^{\mathcal{I}^*} = a^{\mathcal{I}_1^\omega},$$
$$A^{\mathcal{I}^*} = A^{\mathcal{I}_1^\omega}, \text{ if } A \in \Sigma \cup \mathsf{sig}(\mathcal{T}_1), \quad A^{\mathcal{I}^*} = \{f(x) \mid x \in A^{\mathcal{I}_2^\omega}\}, \text{ if } A \notin \Sigma \cup \mathsf{sig}(\mathcal{T}_1),$$
$$P^{\mathcal{I}^*} = P^{\mathcal{I}_1^\omega}, \text{ if } P \in \Sigma \cup \mathsf{sig}(\mathcal{T}_1), \quad P^{\mathcal{I}^*} = \{(f(x), f(y)) \mid (x,y) \in P^{\mathcal{I}_2^\omega}\}, \text{ if } P \notin \Sigma \cup \mathsf{sig}(\mathcal{T}_1).$$

By definition, $\mathcal{I}^* \sim_\Sigma \mathcal{I}_1^\omega$ (actually, $\mathcal{I}^* \sim_{\Sigma \cup \mathsf{sig}(\mathcal{T}_1)} \mathcal{I}_1^\omega$). So $\mathcal{I}^*$ realises the same $\Sigma Q_{\mathcal{T}_1 \cup \mathcal{T}_2}$-types as $\mathcal{I}_1$ and $\mathcal{I}_2$, and $\mathcal{I}^* \models \mathcal{T}_1$. Let us show that $\mathcal{I}^* \models \mathcal{T}_2$. By the definition, we have $x \in A_i^{\mathcal{I}_2^\omega}$ iff $f(x) \in A_i^{\mathcal{I}^*}$, for all points x in $\mathcal{I}_2^\omega$ and all concept names $A_i \in \mathsf{sig}(\mathcal{T}_2)$. As $\mathcal{I}_2^\omega$ is a model for $\mathcal{T}_2$, it is enough to prove that, for every $q \in Q_{\mathcal{T}_1 \cup \mathcal{T}_2}$ and every role $R \in \mathsf{sig}(\mathcal{T}_2)$, we have

$$x \in (\geq q\, R)^{\mathcal{I}_2^\omega} \qquad \text{iff} \qquad f(x) \in (\geq q\, R)^{\mathcal{I}^*}. \tag{1}$$

If R is not a Σ-role then, by the definition above, the number of R-successors of x in $\mathcal{I}_2^\omega$ and $f(x)$ in $\mathcal{I}^*$ is the same. And if R is a Σ-role, then the $\Sigma Q_{\mathcal{T}_1 \cup \mathcal{T}_2}$-type realised by x in $\mathcal{I}_2^\omega$ coincides with the $\Sigma Q_{\mathcal{T}_1 \cup \mathcal{T}_2}$-type realised by $f(x)$ in $\mathcal{I}^*$, which again gives (1). Thus, $\mathcal{I}^* \models \mathcal{T}_1 \cup \mathcal{T}_2$.

We are now in a position to prove Theorem 3 on uniform interpolation. Let $\mathcal{L}$ be either *DL-Lite*$_{bool}$ or *DL-Lite*$_{horn}$, and Σ a signature. We call an $\mathcal{L}$-concept C over Σ a $\Sigma Q^{\mathcal{L}}$*-concept* if all its numerical parameters are in Q. It should be clear that, for every set $\mathcal{S}$ of concept inclusions $C_1 \sqsubseteq C_2$, where C_1, C_2 are $\Sigma Q^{\mathcal{L}}$-concepts, for some finite Q, there is a *finite* set $\mathcal{S}'$ such that $\mathcal{S} \models C \sqsubseteq D$ iff $\mathcal{S}' \models C \sqsubseteq D$ for all concepts C and D.

Theorem 3. Let $\mathcal{T}$ be an $\mathcal{L}$-TBox, Σ a signature, and let $\mathcal{T}'$ be a finite presentation of the set

$$\mathcal{S} = \{C_1 \sqsubseteq C_2 \mid \mathcal{T} \models C_1 \sqsubseteq C_2,\ C_1, C_2\ \Sigma Q^{\mathcal{L}}_{\mathcal{T}}\text{-concepts}\}.$$

Then $\mathcal{T}'$ is a uniform interpolant for $\mathcal{T}$ w.r.t. Σ in $\mathcal{L}$.

Proof. It is sufficient to show that $\mathcal{S} \models C_1 \sqsubseteq C_2$ whenever $\mathcal{T} \models C_1 \sqsubseteq C_2$, for all $\mathcal{L}$-concepts C_1, C_2 with $\mathsf{sig}(C_1 \sqsubseteq C_2) \cap \mathsf{sig}(\mathcal{T}) \subseteq \Sigma$. Suppose, on the contrary, that $\mathcal{S} \not\models C_1 \sqsubseteq C_2$. Take a model $\mathcal{I}_1$ for $\mathcal{S}$ satisfying $C_1 \sqcap \neg C_2$ and realising all $\mathcal{S}$-consistent $\Sigma Q_{\mathcal{T}}$-types. Take a model $\mathcal{I}_2$ for $\mathcal{T}$ and realising all $\mathcal{T}$-consistent $\Sigma Q_{\mathcal{T}}$-types. Then $\mathcal{I}_1$ and $\mathcal{I}_2$ realise exactly the same $\Sigma Q_{\mathcal{T}}$-types. This is clear for $\mathcal{L} = DL\text{-}Lite_{bool}$, and for $\mathcal{L} = DL\text{-}Lite_{horn}$ this follows from the proof of Lemma 7 below. Finally, we use Lemma 2 according to which there exists a model $\mathcal{I}^*$ for $\mathcal{T}$ satisfying $C_1 \sqcap \neg C_2$, which is a contradiction.

We will be using one more immediate consequence of Lemma 2:

Lemma 3. *Let $\mathcal{J}$ be an (at most countable) model for $\mathcal{T}_1$ and Σ a signature with $\Sigma \subseteq \mathsf{sig}(\mathcal{T}_1)$. Suppose that there is a model for $\mathcal{T}_1 \cup \mathcal{T}_2$ realising exactly the same $\Sigma Q_{\mathcal{T}_1 \cup \mathcal{T}_2}$-types as $\mathcal{J}$. Then there is a model $\mathcal{I}^*$ for $\mathcal{T}_1 \cup \mathcal{T}_2$ such that $\mathcal{I}^* \sim_\Sigma \mathcal{J}^\omega$.*

In particular, $\mathcal{I}^* \models \mathcal{A}$ iff $\mathcal{J} \models \mathcal{A}$, for all ABoxes $\mathcal{A}$ over Σ, $\mathcal{I}^* \models \mathcal{T}$ iff $\mathcal{J} \models \mathcal{T}$, for all TBoxes $\mathcal{T}$ over Σ, and $\mathcal{I}^* \models q(\boldsymbol{a})$ iff $\mathcal{J} \models q(\boldsymbol{a})$, for all queries $q(\boldsymbol{a})$ over Σ.

B Model-theoretic characterisations of conservativity

In this section, we give model-theoretic characterisations of the notions of conservativity introduced above. The equivalences stated in Theorem 1 and Theorem 6 will follow from those characterizations. Moreover, they will be used for proving the complexity results stated Theorem 1 and Theorem 6.

B.1 *DL-Lite$_{bool}$*

The following lemma will be used to prove the complexity results stated in Theorem 1. Roughly speaking, it states that whenever $\mathcal{T}_1 \cup \mathcal{T}_2$ is not a deductive conservative extension of $\mathcal{T}_1$ w.r.t. Σ, then there exists a concept over the signature Σ *using only numerical parameters which occur in* $\mathcal{T}_1 \cup \mathcal{T}_2$ which is $\mathcal{T}_1$-consistent but not $\mathcal{T}_1 \cup \mathcal{T}_2$-consistent. For each such concept there obviously exists an equivalent concept which is of polynomial size in $|\Sigma| + |\mathcal{T}_1 \cup \mathcal{T}_2|$, from which one can then easily obtain the complexity upper bound.

In what follows, when considering the problem whether $\mathcal{T}_1 \cup \mathcal{T}_2$ is a conservative extension (according to one of the definitions introduced above) of $\mathcal{T}_1$ w.r.t. Σ, we will always assume that $\Sigma \subseteq \mathsf{sig}(\mathcal{T}_1)$. This is justified by the observation that otherwise we can always add trivial concept implications to $\mathcal{T}_1$ containing all concepts from Σ.

Lemma 4. *The following conditions are equivalent for DL-Lite$_{bool}$ TBoxes $\mathcal{T}_1$, $\mathcal{T}_2$ and a signature $\Sigma \subseteq \mathsf{sig}(\mathcal{T}_1)$:*

(dc) *$\mathcal{T}_1 \cup \mathcal{T}_2$ is a deductive conservative extension of $\mathcal{T}_1$ relative to Σ;*
(mt) *for every countable model $\mathcal{I}$ for $\mathcal{T}_1$, there is a model $\mathcal{I}'$ for $\mathcal{T}_1 \cup \mathcal{T}_2$ such that both $\mathcal{I}'$ and the interpretation $\mathcal{J}_\mathcal{I}$ defined below realise precisely the same $\Sigma Q_{\mathcal{T}_1 \cup \mathcal{T}_2}$-types:*

$$\mathcal{J}_\mathcal{I} \quad = \quad \mathcal{I} \quad \oplus \quad \bigoplus_{\boldsymbol{t} \in \Theta} \mathcal{J}_{\mathcal{T}_1}(\boldsymbol{t}),$$

where Θ is the set of all $\mathcal{T}_1$-consistent $\Sigma Q_{\mathcal{T}_1 \cup \mathcal{T}_2}$-types and $\mathcal{J}_{\mathcal{T}_1}(\boldsymbol{t})$ is a countable model for $\mathcal{T}_1$ realising $\boldsymbol{t}$;
(tp) *every $\mathcal{T}_1$-consistent $\Sigma Q_{\mathcal{T}_1 \cup \mathcal{T}_2}$-type is also $\mathcal{T}_1 \cup \mathcal{T}_2$-consistent.*

Proof. **(mt)** $\Rightarrow$ **(dc)** Suppose otherwise. Then we have $C_1 \sqsubseteq C_2$ with $\mathsf{sig}(C_1 \sqsubseteq C_2) \subseteq \Sigma$ such that $\mathcal{T}_1 \cup \mathcal{T}_2 \models C_1 \sqsubseteq C_2$ and $\mathcal{T}_1 \not\models C_1 \sqsubseteq C_2$. It follows that there is a model $\mathcal{I}$ for $\mathcal{T}_1$ satisfying $C_1 \sqcap \neg C_2$. Clearly, $C_1 \sqcap \neg C_2$ is satisfied in $\mathcal{J}_\mathcal{I}$. Then we apply Lemma 3 to $\mathcal{J}_\mathcal{I}$ and find a model $\mathcal{I}^*$ for $\mathcal{T}_1 \cup \mathcal{T}_2$ satisfying, in particular, $C_1 \sqcap \neg C_2$, which is a contradiction.

(tp) $\Rightarrow$ **(mt)** For every $\mathcal{T}_1$-consistent $\Sigma Q_{\mathcal{T}_1 \cup \mathcal{T}_2}$ type $\boldsymbol{t}$, we take a model $\mathcal{I}'_{\boldsymbol{t}}$ for $\mathcal{T}_1 \cup \mathcal{T}_2$ realising $\boldsymbol{t}$. Then the disjoint union of all such $\mathcal{I}'_{\boldsymbol{t}}$ realises precisely the same set of $\Sigma Q_{\mathcal{T}_1 \cup \mathcal{T}_2}$-types as $\mathcal{I}_\mathcal{J}$.

(dc) $\Rightarrow$ **(tp)** Suppose that there is a $\Sigma Q_{\mathcal{T}_1 \cup \mathcal{T}_2}$-type $\boldsymbol{t}$ that is $\mathcal{T}_1$-consistent but not $\mathcal{T}_1 \cup \mathcal{T}_2$-consistent. Then $\mathcal{T}_1 \cup \mathcal{T}_2 \models \bigsqcap_{C \in \boldsymbol{t}} C \sqsubseteq \bot$ but $\mathcal{T}_1 \not\models \bigsqcap_{C \in \boldsymbol{t}} C \sqsubseteq \bot$, contrary to $\mathcal{T}_1 \cup \mathcal{T}_2$ being a deductive conservative extension of $\mathcal{T}_1$.

Let us now consider the other types of conservativity for *DL-Lite$_{bool}$* TBoxes. The equivalences stated in Theorem 6 for *DL-Lite$_{bool}$* follow from the next lemma. Moreover, again it follows that only parameters occurring in $\mathcal{T}_1 \cup \mathcal{T}_1$ have to be considered.

Lemma 5. *The following conditions are equivalent for DL-Lite$_{bool}$ TBoxes $\mathcal{T}_1$, $\mathcal{T}_2$ and a signature $\Sigma \subseteq \mathsf{sig}(\mathcal{T}_1)$:*

(sdc) *$\mathcal{T}_1 \cup \mathcal{T}_2$ is a strong deductive conservative extension of $\mathcal{T}_1$ relative to Σ;*
(qc) *$\mathcal{T}_1 \cup \mathcal{T}_2$ is a query conservative extension of $\mathcal{T}_1$ relative to Σ;*
(sqc) *$\mathcal{T}_1 \cup \mathcal{T}_2$ is a strong query conservative extension of $\mathcal{T}_1$ relative to Σ;*
(mt) *for every model $\mathcal{I}$ for $\mathcal{T}_1$, there is a model $\mathcal{I}'$ for $\mathcal{T}_1 \cup \mathcal{T}_2$ such that $\mathcal{I}'$ and $\mathcal{I}$ realise precisely the same $\Sigma Q_{\mathcal{T}_1 \cup \mathcal{T}_2}$-types;*
(tp) *if a set of $\Sigma Q_{\mathcal{T}_1 \cup \mathcal{T}_2}$-types is precisely $\mathcal{T}_1$-realisable, then it is precisely $\mathcal{T}_1 \cup \mathcal{T}_2$-realisable.*

Proof. Implications **(sqc)** $\Rightarrow$ **(qc)** and **(sqc)** $\Rightarrow$ **(sdc)** and **(tp)** $\Rightarrow$ **(mt)** follow immediately from definitions.

(mt) $\Rightarrow$ **(sqc)** Suppose otherwise. Then we have a TBox $\mathcal{T}$, an ABox $\mathcal{A}$, and a query $q(\boldsymbol{x})$ all over Σ such that $(\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}, \mathcal{A}) \models q(\boldsymbol{a})$ but $(\mathcal{T}_1 \cup \mathcal{T}, \mathcal{A}) \not\models q(\boldsymbol{a})$, for some $\boldsymbol{a}$. Take a model $\mathcal{I}$ for $(\mathcal{T}_1 \cup \mathcal{T}, \mathcal{A})$ such that $\mathcal{I} \not\models q(\boldsymbol{a})$. By Lemma 3, there is a model $\mathcal{I}^*$ for $\mathcal{T}_1 \cup \mathcal{T}_2$ such that $\mathcal{I}^* \models (\mathcal{T}, \mathcal{A})$ and $\mathcal{I}^* \not\models q(\boldsymbol{a})$, which is a contradiction.

(qc) $\Rightarrow$ **(tp)**. Let Ξ be a precisely $\mathcal{T}_1$-realisable set of $\Sigma Q_{\mathcal{T}_1 \cup \mathcal{T}_2}$-types and suppose that it is not precisely $\mathcal{T}_1 \cup \mathcal{T}_2$-realisable. Then two cases are possible:

1. For every model $\mathcal{I}$ of $\mathcal{T}_1 \cup \mathcal{T}_2$ there is some $\boldsymbol{t} \in \Xi$ that is not realised in $\mathcal{I}$. Consider the ABox $\mathcal{A} = \{C(a_{\boldsymbol{t}}) \mid C \in \boldsymbol{t},\ \boldsymbol{t} \in \Xi\}$ and the query
$$q = \bot.$$
We then have $(\mathcal{T}_1 \cup \mathcal{T}_2, \mathcal{A}) \models q$ but $(\mathcal{T}_1, \mathcal{A}) \not\models q$, which is a contradiction.
2. Suppose now that Case 1 does not hold. For each $\boldsymbol{t} \in \Xi$, take an object name $a_{\boldsymbol{t}}$. Consider the ABox $\mathcal{A} = \{C(a_{\boldsymbol{t}}) \mid C \in \boldsymbol{t}, \boldsymbol{t} \in \Xi\}$. Let Θ be the set of all $\mathcal{T}_1 \cup \mathcal{T}_2$-realisable $\Sigma Q_{\mathcal{T}_1 \cup \mathcal{T}_2}$-types that are not in Ξ. Now consider the query
$$q = \exists x \bigvee_{\boldsymbol{t} \in \Theta} \bigwedge_{C \in \boldsymbol{t}} C(x).$$
Then $(\mathcal{T}_1 \cup \mathcal{T}_2, \mathcal{A}) \models q$ but $(\mathcal{T}_1, \mathcal{A}) \not\models q$, which is again a contradiction.

(sdc) $\Rightarrow$ **(tp)**. Let Ξ be a set of precisely $\mathcal{T}_1$-realisable $\Sigma Q_{\mathcal{T}_1 \cup \mathcal{T}_2}$-types, and take
$$\mathcal{T}_\Xi = \{\top \sqsubseteq \bigsqcup_{\boldsymbol{t} \in \Xi} \bigsqcap_{C \in \boldsymbol{t}} C\}.$$
Clearly, $\mathcal{T}_1 \cup \mathcal{T}_\Xi \not\models \bigsqcap_{C \in \boldsymbol{t}} C \sqsubseteq \bot$, for every $\boldsymbol{t} \in \Xi$. Then, by **(sdc)**, $\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_\Xi \not\models \bigsqcap_{C \in \boldsymbol{t}} C \sqsubseteq \bot$ and thus, there is a model $\mathcal{I}_{\boldsymbol{t}}$ for $\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_\Xi$ realising $\boldsymbol{t}$. Take the disjoint union $\mathcal{I}$ of all these models $\mathcal{I}_{\boldsymbol{t}}$. It is easy to see that $\mathcal{I}$ is a model for $\mathcal{T}_1 \cup \mathcal{T}_2$ realising precisely the types Ξ.

B.2 *DL-Lite*$_{horn}$

By a ΣQ^h-*concept* we mean any concept of the form $\bot$, A_i or $\geq q\, R$, for some $A_i \in \Sigma$, Σ-role R and $q \in Q$. A ΣQ^h *concept inclusion* is of the form $B_1 \sqcap \dots \sqcap B_k \sqsubseteq B$, where $B_1, \dots, B_k, B$ are ΣQ^h-concepts. In what follows, an empty conjunction $\bigsqcap_{i \in \emptyset} B_i$ stands for $\top$.

Given a ΣQ-type $\boldsymbol{t}$, let

$$\boldsymbol{t}^+ = \{B \in \boldsymbol{t} \mid B \text{ a basic concept}\} \qquad \text{and} \qquad \boldsymbol{t}^- = \{\neg B \in \boldsymbol{t} \mid B \text{ a basic concept}\}.$$

Say that a ΣQ-type $\boldsymbol{t}_1$ is *h-contained* in a ΣQ-type $\boldsymbol{t}_2$ if $\boldsymbol{t}_1^+ \subseteq \boldsymbol{t}_2^+$.

Given a Horn TBox $\mathcal{T}$ and a ΣQ-type with $\Sigma \subseteq \mathsf{sig}(\mathcal{T})$ and $Q \subseteq Q_{\mathcal{T}}$, define the $\mathcal{T}$-*closure* $\mathsf{cl}_{\mathcal{T}}(\boldsymbol{t})$ of $\boldsymbol{t}$ as the $\mathsf{sig}(\mathcal{T})Q_{\mathcal{T}}$-type $\mathsf{cl}_{\mathcal{T}}(\boldsymbol{t})$ such that

$$\mathsf{cl}_{\mathcal{T}}(\boldsymbol{t})^+ = \{B \text{ a basic } \mathsf{sig}(\mathcal{T})Q_{\mathcal{T}}\text{-concept} \mid \mathcal{T} \models \bigsqcap_{B_k \in \boldsymbol{t}^+} B_k \sqsubseteq B\}.$$

As we will see later in Theorem 10, $\mathsf{cl}_{\mathcal{T}}(\boldsymbol{t})$ can be computed in polynomial time in the size of $\mathcal{T}$. Moreover, we have the following:

Lemma 6. *Let $\mathcal{T}$ be a DL-Lite$_{horn}$-TBox. A ΣQ-type $\boldsymbol{t}$ is $\mathcal{T}$-consistent iff $\boldsymbol{t} = \mathsf{cl}_{\mathcal{T}}(\boldsymbol{t}) \restriction \Sigma Q$ and $\bot \notin \boldsymbol{t}$, where $\restriction \Sigma Q$ means restriction to ΣQ-types. Moreover, $\mathcal{T}$ enjoys the 'disjunction property:' if $\mathcal{T} \models \bigsqcap B'_j \sqsubseteq \bigsqcup B_i$, where the B'_j and B_i are basic concepts, then there is some i such that $\mathcal{T} \models \bigsqcap B'_j \sqsubseteq B_i$.*

We are now in a position to prove the first part of Theorem 1.

Lemma 7. *For any DL-Lite$_{horn}$ TBoxes $\mathcal{T}_1$, $\mathcal{T}_2$ and any signature Σ, the following two conditions are equivalent:*

- *$\mathcal{T}_1 \cup \mathcal{T}_2$ is a conservative extension of $\mathcal{T}_1$ in DL-Lite$_{bool}$ w.r.t Σ;*
- *$\mathcal{T}_1 \cup \mathcal{T}_2$ is a conservative extension of $\mathcal{T}_1$ in DL-Lite$_{horn}$ w.r.t. Σ.*

Proof. Suppose that $\mathcal{T}_1 \cup \mathcal{T}_2$ is not a conservative extension of $\mathcal{T}_1$ in *DL-Lite$_{bool}$* w.r.t. Σ (where again, without loss of generality, we assume that $\Sigma \subseteq \mathsf{sig}(\mathcal{T}_1)$). By Lemma 4, there exists a $\mathcal{T}_1$-consistent $\Sigma Q_{\mathcal{T}_1 \cup \mathcal{T}_2}$-type $\boldsymbol{t}$ which is not $\mathcal{T}_1 \cup \mathcal{T}_2$-consistent. Consider now the $\mathcal{T}_1$ and $\mathcal{T}_1 \cup \mathcal{T}_2$-closures $\mathsf{cl}_{\mathcal{T}_1}(\boldsymbol{t})$ and $\mathsf{cl}_{\mathcal{T}_1 \cup \mathcal{T}_2}(\boldsymbol{t})$ of $\boldsymbol{t}$. Since $\boldsymbol{t}$ is $\mathcal{T}_1$-consistent, we have $\mathsf{cl}_{\mathcal{T}_1}(\boldsymbol{t}) \restriction \Sigma Q_{\mathcal{T}_1 \cup \mathcal{T}_2} = \boldsymbol{t}$ by Lemma 6; on the other hand, as $\boldsymbol{t}$ is not $\mathcal{T}_1 \cup \mathcal{T}_2$-consistent, $\boldsymbol{t}$ is properly h-contained in $\mathsf{cl}_{\mathcal{T}_1 \cup \mathcal{T}_2}(\boldsymbol{t}) \restriction \Sigma Q_{\mathcal{T}_1 \cup \mathcal{T}_2}$, again by Lemma 6. Therefore, there is $B \in (\mathsf{cl}_{\mathcal{T}_1 \cup \mathcal{T}_2}(\boldsymbol{t}) \setminus \mathsf{cl}_{\mathcal{T}_1}(\boldsymbol{t})) \restriction \Sigma Q_{\mathcal{T}_1 \cup \mathcal{T}_2}$ such that

$$\mathcal{T}_1 \not\models \bigsqcap_{B_k \in \boldsymbol{t}^+} B_k \sqsubseteq B \qquad \text{and} \qquad \mathcal{T}_1 \cup \mathcal{T}_2 \models \bigsqcap_{B_k \in \boldsymbol{t}^+} B_k \sqsubseteq B,$$

which means that $\mathcal{T}_1 \cup \mathcal{T}_2$ is not a conservative extension in *DL-Lite$_{horn}$* of $\mathcal{T}_1$ w.r.t. Σ.

Let $\mathcal{I}$ and $\mathcal{I}'$ be interpretations and Σ a signature. A map f from $\Delta^{\mathcal{I}}$ into $\Delta^{\mathcal{I}'}$ is called a *Σ-homomorphism* if the following conditions hold for all $x, y \in \Delta^{\mathcal{I}}$:

- $f(a^{\mathcal{I}}) = a^{\mathcal{I}'}$, for every object name a,
- $x \in A^{\mathcal{I}}$ implies $f(x) \in A^{\mathcal{I}'}$, for every concept name A in Σ;
- $(x, y) \in P^{\mathcal{I}}$ implies $(f(x), f(y)) \in P^{\mathcal{I}'}$, for every role name P in Σ.

The following lemma gives the main model-theoretic property characterising *DL-Lite$_{horn}$* TBoxes:

Lemma 8. *Let $\mathcal{T}$ be a DL-Lite$_{horn}$ TBox and $\boldsymbol{t}$ a $\mathcal{T}$-realisable ΣQ-type. Then there exists (at most countable) model $\mathcal{J}_{\mathcal{T}}(\boldsymbol{t})$ for $\mathcal{T}$ such that $\mathcal{J}_{\mathcal{T}}(\boldsymbol{t})$ realises $\boldsymbol{t}$ and, for every model $\mathcal{I}$ for $\mathcal{T}$ realising $\boldsymbol{t}$, there exists a Σ-homomorphism $h\colon \mathcal{J}_{\mathcal{T}}(\boldsymbol{t}) \to \mathcal{I}$.*

In what follows we fix one model $\mathcal{J}_{\mathcal{T}}(\boldsymbol{t})$ with the properties mentioned above and call it call the *minimal model for $\mathcal{T}$ realising* $\boldsymbol{t}$. As an immediate consequence of Lemma 8 we obtain the following:

Lemma 9. *Let $\mathcal{T}$ be a DL-Lite$_{horn}$ TBox and $\boldsymbol{t}$ a $\mathcal{T}$-realisable ΣQ-type. Then there exists (at most countable) model $\mathcal{J}_{\mathcal{T}}(\boldsymbol{t})$ for $\mathcal{T}$ such that $\mathcal{J}_{\mathcal{T}}(\boldsymbol{t})$ realises $\boldsymbol{t}$ and, for every model $\mathcal{I}$ for $\mathcal{T}$ realising $\boldsymbol{t}$, there is a Σ-homomorphism $h\colon \mathcal{I} \oplus \mathcal{J}_{\mathcal{T}}(\boldsymbol{t}) \to \mathcal{I}$. In particular,*

- *all ΣQ-types that are realised in $\mathcal{I}$ are also realised in $\mathcal{I} \oplus \mathcal{J}_{\mathcal{T}}(\boldsymbol{t})$;*
- *every ΣQ-type realised in $\mathcal{I} \oplus \mathcal{J}_{\mathcal{T}}(\boldsymbol{t})$ is h-contained in some ΣQ-type realised in $\mathcal{I}$;*

– *if* $\mathcal{I} \oplus \mathcal{J}_{\mathcal{T}}(\boldsymbol{t}) \models q(\boldsymbol{a})$ *then* $\mathcal{I} \models q(\boldsymbol{a})$, *for every positive existential* (= *DL-Lite*$_{horn}$) *query* $q(\boldsymbol{a})$ *over* Σ.

A set Ξ of ΣQ-types is said to be *sub-precisely* $\mathcal{T}$*-realisable* if there is a model $\mathcal{I}$ for $\mathcal{T}$ such that $\mathcal{I}$ realises all types from Ξ, and every ΣQ-type realised in $\mathcal{I}$ is h-contained in some type from Ξ.

Lemma 10. *For any DL-Lite*$_{horn}$ *TBoxes* $\mathcal{T}_1$, $\mathcal{T}_2$ *and any signature* Σ, *the following conditions are equivalent:*

(qc) $\mathcal{T}_1 \cup \mathcal{T}_2$ *is a query conservative extension of* $\mathcal{T}_1$ *in DL-Lite*$_{horn}$ *w.r.t.* Σ*;*
(mt) *for every model* $\mathcal{I}$ *for* $\mathcal{T}_1$, *there exists a model* $\mathcal{I}'$ *for* $\mathcal{T}_1 \cup \mathcal{T}_2$ *such that both* $\mathcal{I}'$ *and the interpretation* $\mathcal{J}_{\mathcal{I}}$ *defined below realise precisely the same* $\Sigma Q_{\mathcal{T}_1 \cup \mathcal{T}_2}$*-types:*

$$\mathcal{J}_{\mathcal{I}} \quad = \quad \mathcal{I} \quad \oplus \quad \bigoplus_{\boldsymbol{t} \in \Theta_{\mathcal{I}}} \mathcal{J}_{\mathcal{T}_1}(\boldsymbol{t}),$$

where $\Theta_{\mathcal{I}}$ *is the set of all* $\mathcal{T}_1$*-consistent* $\Sigma Q_{\mathcal{T}_1 \cup \mathcal{T}_2}$*-types that are h-contained in some types realised in* $\mathcal{I}$*;*
(tp) *if a set of* $\Sigma Q_{\mathcal{T}_1 \cup \mathcal{T}_2}$*-types is precisely* $\mathcal{T}_1$*-realisable, then it is sub-precisely* $\mathcal{T}_1 \cup \mathcal{T}_2$*-realisable.*

Proof. **(mt)** $\Rightarrow$ **(qc)**. Suppose otherwise. Then there are an ABox $\mathcal{A}$ and a query $q(\boldsymbol{a})$ over Σ such that $(\mathcal{T}_1 \cup \mathcal{T}_2, \mathcal{A}) \models q(\boldsymbol{a})$ but $(\mathcal{T}_1, \mathcal{A}) \not\models q(\boldsymbol{a})$. Let $\mathcal{I}$ be a model for $(\mathcal{T}_1, \mathcal{A})$ with $\mathcal{I} \not\models q(\boldsymbol{a})$. By Lemma 9, $\mathcal{J}_{\mathcal{I}} \not\models q(\boldsymbol{a})$. But then we can apply Lemma 3 to $\mathcal{J}_{\mathcal{I}}$ and $\mathcal{I}'$, and find a model $\mathcal{I}^*$ for $(\mathcal{T}_1 \cup \mathcal{T}_2, \mathcal{A})$ such that $\mathcal{I}^* \not\models q(\boldsymbol{a})$, which is a contradiction.

(tp) $\Rightarrow$ **(mt)**. Let $\mathcal{I}$ be a model for $\mathcal{T}_1$ and Ξ the set of $\Sigma Q_{\mathcal{T}_1 \cup \mathcal{T}_2}$-types realised in $\mathcal{J}_{\mathcal{I}}$. By **(tp)**, there exists a model $\mathcal{I}'$ for $\mathcal{T}_1 \cup \mathcal{T}_2$ such that it realises all types in Ξ and every $\Sigma Q_{\mathcal{T}_1 \cup \mathcal{T}_2}$-type realised in it is h-contained in some type from Ξ. But then clearly $\mathcal{I}'$ and $\mathcal{J}_{\mathcal{I}}$ realise precisely the same $\Sigma Q_{\mathcal{T}_1 \cup \mathcal{T}_2}$-types.

(qc) $\Rightarrow$ **(tp)**. Suppose that **(tp)** does not hold for some Ξ. Then two cases are possible:

1. For every model $\mathcal{I}$ for $\mathcal{T}_1 \cup \mathcal{T}_2$, there is $\boldsymbol{t} \in \Xi$ such that $(\bigsqcap_{B \in \boldsymbol{t}^+} B)^{\mathcal{I}} = \emptyset$. Consider the ABox $\mathcal{A} = \{B(a_{\boldsymbol{t}}) \mid \boldsymbol{t} \in \Xi, \quad B \in \boldsymbol{t}^+\}$, where $a_{\boldsymbol{t}}$, for $\boldsymbol{t} \in \Xi$, are object names, and the query
$$q = \bot.$$
Then $(\mathcal{T}_1, \mathcal{A}) \not\models q$, while $(\mathcal{T}_1 \cup \mathcal{T}_2, \mathcal{A}) \models q$.
2. Case 1 does not hold. Then, for every model $\mathcal{I}'$ for $\mathcal{T}_1 \cup \mathcal{T}_2$, there is a $\Sigma Q_{\mathcal{T}_1 \cup \mathcal{T}_2}$-type $\boldsymbol{t}$ which is realised in $\mathcal{I}'$ and not h-contained in any type from Ξ. Let Θ be the set of all such $\Sigma Q_{\mathcal{T}_1 \cup \mathcal{T}_2}$-types. Now consider the ABox $\mathcal{A} = \{B(a_{\boldsymbol{t}}) \mid \boldsymbol{t} \in \Xi, \quad B \in \boldsymbol{t}^+\}$, where $a_{\boldsymbol{t}}$, for $\boldsymbol{t} \in \Xi$, are object names, and the query
$$q \quad = \quad \exists x \bigvee_{\boldsymbol{t} \in \Theta} \bigwedge_{B \in \boldsymbol{t}^+} B(x).$$
We then have $(\mathcal{T}_1, \mathcal{A}) \not\models q$ but $(\mathcal{T}_1 \cup \mathcal{T}_2, \mathcal{A}) \models q$.

Finally, we show the equivalence of (2) and (3) in Theorem 6 for *DL-Lite*$_{horn}$.

Given a consistent set Ξ of ΣQ-types, define the TBox $\mathcal{T}_\Xi$ *induced by* Ξ by taking

$$\mathcal{T}_\Xi = \big\{B_1 \sqcap \cdots \sqcap B_k \sqsubseteq B \mid B_1, \dots, B_k, B \text{ are distinct } \Sigma Q^h\text{-concepts such that, for all } \boldsymbol{t} \in \Xi, \text{ if } B_1, \dots, B_k \in \boldsymbol{t}^+ \text{ then } B \in \boldsymbol{t}^+\big\}.$$

Note that (i) if, for distinct ΣQ^h-concepts $B_1, \dots, B_k$, there is no $\boldsymbol{t} \in \Xi$ with $B_1, \dots, B_k \in \boldsymbol{t}^+$ then $B_1 \sqcap \cdots \sqcap B_k \sqsubseteq \bot$ is in $\mathcal{T}_\Xi$, and (ii) if $B \in \boldsymbol{t}^+$, for all $\boldsymbol{t} \in \Xi$, then $\top \sqsubseteq B$ is in $\mathcal{T}_\Xi$.

Lemma 11. *Let Ξ be a set of ΣQ-types and $\boldsymbol{t}_0$ a ΣQ-type. Let $\Lambda_{\boldsymbol{t}_0} = \{\boldsymbol{t} \in \Xi \mid \boldsymbol{t}_0^+ \subseteq \boldsymbol{t}^+\}$. Then $\boldsymbol{t}_0$ is $\mathcal{T}_\Xi$-consistent iff $\Lambda_{\boldsymbol{t}_0} \neq \emptyset$ and $\boldsymbol{t}_0^+ = \bigcap_{\boldsymbol{t} \in \Lambda_{\boldsymbol{t}_0}} \boldsymbol{t}^+$.*

Proof. ($\Rightarrow$) Clearly, if $\Lambda_{\boldsymbol{t}_0} = \emptyset$ then $\bigsqcap_{B \in \boldsymbol{t}_0^+} B \sqsubseteq \bot$ is in $\mathcal{T}_\Xi$, and so $\boldsymbol{t}_0$ cannot be $\mathcal{T}_\Xi$-consistent. If $\Lambda_{\boldsymbol{t}_0} \neq \emptyset$ then, for every $B' \in \bigcap_{\boldsymbol{t} \in \Lambda_{\boldsymbol{t}_0}} \boldsymbol{t}^+$, we have $(\bigsqcap_{B \in \boldsymbol{t}_0^+} \sqsubseteq B') \in \mathcal{T}_\Xi$. So $\boldsymbol{t}^+ \supseteq \bigcap_{\boldsymbol{t} \in \Lambda_{\boldsymbol{t}_0}} \boldsymbol{t}^+$.

($\Leftarrow$) If there is no model for $\mathcal{T}_\Xi$ realising $\boldsymbol{t}_0$, then we have $\mathcal{T}_\Xi \models \bigsqcap_{B \in \boldsymbol{t}_0^+} B \sqsubseteq \bigsqcup_{\neg B \in \boldsymbol{t}_0^-} B$. But then, by Lemma 6, $\mathcal{T}_\Xi \models \bigsqcap_{B \in \boldsymbol{t}_0^+} \sqsubseteq B'$, for some $\neg B' \in \boldsymbol{t}_0^-$. Therefore, $\bigsqcap_{B \in \boldsymbol{t}_0^+} B \sqsubseteq B' \in \mathcal{T}_\Xi$, and so B' must be in $\boldsymbol{t}^+$, which is impossible.

A set Ξ of ΣQ-types is said to be *join-precisely $\mathcal{T}$-realisable* if there is a model $\mathcal{I}$ for $\mathcal{T} \cup \mathcal{T}_\Xi$ such that $\mathcal{I}$ realises all types from Ξ. (It follows that every ΣQ-type realised in $\mathcal{I}$ is h-contained in a type from Ξ.) As we shall see later, given a type $\boldsymbol{t}$, the $\mathcal{T} \cup \mathcal{T}_\Xi$-closure $\mathsf{cl}_{\mathcal{T} \cup \mathcal{T}_\Xi}(\boldsymbol{t})$ can be computed in polynomial type in the size of $\mathcal{T}$, Ξ and $\boldsymbol{t}$.

Lemma 12. *Let Ξ be the set of ΣQ-types that is precisely realised in a model $\mathcal{I}$ for a TBox $\mathcal{T}$. Then*

1. *$\mathcal{I} \models \mathcal{T}_\Xi$;*
2. *$\mathcal{T} \models \bigsqcap_k B_k \sqsubseteq B$ implies $\mathcal{T}_\Xi \models \bigsqcap_k B_k \sqsubseteq B$ (which holds iff $\bigsqcap_k B_k \sqsubseteq B \in \mathcal{T}_\Xi$), for all ΣQ^h-concept inclusions $\bigsqcap_k B_k \sqsubseteq B$.*

Proof. The first claim is obvious. To show the second one, assume that $\mathcal{T} \models \bigsqcap_k B_k \sqsubseteq B$, but $\mathcal{T}_\Xi \not\models \bigsqcap_k B_k \sqsubseteq B$. Then $\bigsqcap_k B_k \sqsubseteq B \notin \mathcal{T}_\Xi$, and so there is a type $\boldsymbol{t} \in \Xi$ such that $B_k \in \boldsymbol{t}$, while $B \notin \boldsymbol{t}$. But this means that $\mathcal{I} \not\models \bigsqcap_k B_k \sqsubseteq B$, which is impossible, since $\mathcal{I} \models \mathcal{T}$.

Lemma 13. *For any DL-Lite$_{horn}$ TBoxes $\mathcal{T}_1$, $\mathcal{T}_2$ and any signature Σ, the following conditions are equivalent:*

(sdc) *$\mathcal{T}_1 \cup \mathcal{T}_2$ is a strong deductive conservative extension of $\mathcal{T}_1$ in DL-Lite$_{horn}$ w.r.t. Σ;*

(sqc) *$\mathcal{T}_1 \cup \mathcal{T}_2$ is a strong query conservative extension of $\mathcal{T}_1$ in DL-Lite$_{horn}$ w.r.t. Σ;*

(mt) *for every model $\mathcal{I}$ for $\mathcal{T}_1$, which precisely realises a set Ξ of $\Sigma Q_{\mathcal{T}_1\cup\mathcal{T}_2}$-types, there exists a model $\mathcal{I}'$ for $\mathcal{T}_1 \cup \mathcal{T}_2$ such that both $\mathcal{I}'$ and the interpretation $\mathcal{J}_{\mathcal{I}}$ defined below realise precisely the same $\Sigma Q_{\mathcal{T}_1\cup\mathcal{T}_2}$-types:*

$$\mathcal{J}_{\mathcal{I}} \quad = \quad \mathcal{I} \quad \oplus \quad \bigoplus_{\boldsymbol{t}\in\Theta_{\mathcal{I}}} \mathcal{J}_{\mathcal{T}_1\cup\mathcal{T}_\Xi}(\boldsymbol{t}),$$

where $\Theta_{\mathcal{I}}$ is the set of all $\mathcal{T}_1 \cup \mathcal{T}_\Xi$-consistent $\Sigma Q_{\mathcal{T}_1\cup\mathcal{T}_2}$-types;

(tp) *if a set of $\Sigma Q_{\mathcal{T}_1\cup\mathcal{T}_2}$-types is precisely $\mathcal{T}_1$-realisable, then it is join-precisely $\mathcal{T}_1 \cup \mathcal{T}_2$-realisable.*

Proof. The implication **(sqc)** $\Rightarrow$ **(sdc)** is trivial.

(mt) $\Rightarrow$ **(sqc)**. Suppose otherwise. Let $\mathcal{T}$ be a TBox, $\mathcal{A}$ an ABox and $q(\boldsymbol{x})$ a query over Σ such that $(\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}, \mathcal{A}) \models q(\boldsymbol{a})$ but $(\mathcal{T}_1 \cup \mathcal{T}, \mathcal{A}) \not\models q(\boldsymbol{a})$. Let $\mathcal{I}$ be a model of $(\mathcal{T}_1 \cup \mathcal{T}, \mathcal{A})$ with $\mathcal{I} \not\models q(\boldsymbol{a})$. By Lemma 12, $\mathcal{I} \models \mathcal{T}_\Xi$. By Lemma 9, applied to $\mathcal{T}_1 \cup \mathcal{T}_\Xi$, $\mathcal{J}_{\mathcal{I}} \not\models q(\boldsymbol{a})$. But then we can apply Lemma 3 to $\mathcal{J}_{\mathcal{I}}$ and $\mathcal{I}'$ and find a model $\mathcal{I}^*$ for $\mathcal{T}_1 \cup \mathcal{T}_2$ that is Σ-isomorphic to $\mathcal{J}_{\mathcal{I}}^\omega$. As $\mathcal{J}_{\mathcal{I}} \models \mathcal{T}$, it follows that $\mathcal{I}^*$ is a model for $(\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}, \mathcal{A})$ such that $\mathcal{I}^* \not\models q(\boldsymbol{a})$, which is a contradiction.

(tp) $\Rightarrow$ **(mt)**. Let $\mathcal{I}$ be a model for $\mathcal{T}_1$ and Ξ the set of $\Sigma Q_{\mathcal{T}_1\cup\mathcal{T}_2}$-types realised in $\mathcal{I}$. Observe that $\mathcal{J}_{\mathcal{I}}$ precisely realises $\Theta_{\mathcal{I}}$: indeed, $\mathcal{J}_{\mathcal{I}}$ realises every $\mathcal{T}_1 \cup \mathcal{T}_\Xi$-consistent $\Sigma Q_{\mathcal{T}_1\cup\mathcal{T}_2}$-type and conversely, every $\Sigma Q_{\mathcal{T}_1\cup\mathcal{T}_2}$-type realised in $\mathcal{J}_{\mathcal{I}}$, is $\mathcal{T}_1 \cup \mathcal{T}_\Xi$-consistent.

By **(tp)**, there exists, for the set $\Theta_{\mathcal{I}}$ of $\Sigma Q_{\mathcal{T}_1\cup\mathcal{T}_2}$-types, a model $\mathcal{I}'$ for $\mathcal{T}_1 \cup \mathcal{T}_2$ such that it realises all types in $\Theta_{\mathcal{I}}$ and each $\Sigma Q_{\mathcal{T}_1\cup\mathcal{T}_2}$-type realised in $\mathcal{I}'$ is $\mathcal{T}_{\Theta_{\mathcal{I}}}$-consistent. We claim that $\mathcal{I}'$ realises precisely the set $\Theta_{\mathcal{I}}$ of $\Sigma Q_{\mathcal{T}_1\cup\mathcal{T}_2}$-types. Indeed, $\mathcal{I}'$ realises every type from $\Theta_{\mathcal{I}}$; conversely, let $\boldsymbol{t}$ be a $\Sigma Q_{\mathcal{T}_1\cup\mathcal{T}_2}$-type realised in $\mathcal{I}'$; as $\boldsymbol{t}$ is $\mathcal{T}_{\Theta_{\mathcal{I}}}$-consistent, by Lemma 11, there are $\boldsymbol{t}_1, \dots, \boldsymbol{t}_k \in \Theta_{\mathcal{I}}$ such that $\boldsymbol{t}^+ = \bigcap_k \boldsymbol{t}_i^+$; as the $\boldsymbol{t}_k$ are all realised in $\mathcal{J}_{\mathcal{I}}$, they are $\mathcal{T}_1 \cup \mathcal{T}_\Xi$-consistent and therefore, $\boldsymbol{t}$ is $\mathcal{T}_1 \cup \mathcal{T}_\Xi$-consistent and $\boldsymbol{t} \in \Theta_{\mathcal{I}}$.

(sdc) $\Rightarrow$ **(tp)**. Let Ξ be a set of precisely $\mathcal{T}_1$-realisable $\Sigma Q_{\mathcal{T}_1\cup\mathcal{T}_2}$-types. We claim that, for all $\Sigma Q_{\mathcal{T}_1\cup\mathcal{T}_2}$ concept inclusions $C \sqsubseteq \bigsqcup B'_k$ with $C = \bigsqcap B_k$,

$$\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_\Xi \models C \sqsubseteq \bigsqcup B'_k \qquad \text{iff} \qquad \mathcal{T}_1 \models C \sqsubseteq \bigsqcup B'_k.$$

Indeed, if $\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_\Xi \models C \sqsubseteq \bigsqcup B'_k$ then, by Lemma 6, there is j with $\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_\Xi \models C \sqsubseteq B'_j$ and, by **(sdc)**, we have $\mathcal{T}_1 \cup \mathcal{T}_\Xi \models C \sqsubseteq B'_j$. Then, by Lemma 12, $\mathcal{T}_1 \models C \sqsubseteq B'_j$, from which the claim follows. The converse implication is obvious.

Clearly, for each $\boldsymbol{t} \in \Xi$, we have

$$\mathcal{T}_1 \not\models \bigsqcap_{B\in\boldsymbol{t}^+} B \sqsubseteq \bigsqcup_{\neg B\in\boldsymbol{t}^-} B,$$

and therefore, there is a model for $\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_\Xi$ realising $\boldsymbol{t}$. Take the disjoint union $\mathcal{I}'$ of all models $\mathcal{I}_{\boldsymbol{t}}$, for $\boldsymbol{t} \in \Xi$. Clearly, $\mathcal{I}'$ realises all types in Ξ and each $\Sigma Q_{\mathcal{T}_1\cup\mathcal{T}_2}$-type $\boldsymbol{t}$ realised in it is Ξ-consistent.

B.3 Robustness

Lemma 14. *All the notions of conservativity considered in this paper are robust under joins and vocabulary extensions, for both DL-Lite$_{bool}$ and DL-Lite$_{horn}$.*

Proof. First we consider the case of query conservativity in *DL-Lite$_{horn}$*. If $\mathcal{T}_i \cup \mathcal{T}_0$ is a query conservative extension of $\mathcal{T}_0$ w.r.t. Σ, for $i = 1, 2$, then by Lemma 10, for every model $\mathcal{I}$ for $\mathcal{T}_0$, there is a model $\mathcal{I}_i$ for $\mathcal{T}_0 \cup \mathcal{T}_i$ such that $\mathcal{I}_i$ and $\mathcal{J}_\mathcal{I}$ realise the same ΣQ-types (for $Q = Q_{\mathcal{T}_0 \cup \mathcal{T}_1 \cup \mathcal{T}_2}$). We may assume that the $\mathcal{I}_i$ is at most countable. Then, by Lemma 2, there is a model $\mathcal{I}^*$ of $\mathcal{T}_0 \cup \mathcal{T}_1 \cup \mathcal{T}_2$ realising the same ΣQ-types as $\mathcal{J}_\mathcal{I}$, which, by Lemma 10, implies that $\mathcal{T}_0 \cup \mathcal{T}_1 \cup \mathcal{T}_2$ is a query conservative extension of $\mathcal{T}_0$.

Strong query conservativity for both *DL-Lite$_{horn}$* and *DL-Lite$_{bool}$* is considered analogously (the only difference is that instead of Lemma 10 we apply Lemma 13 for *DL-Lite$_{horn}$* and Lemma 5 for *DL-Lite$_{bool}$*, respectively).

Deductive conservativity for *DL-Lite$_{bool}$* is also considered analogously (we apply Lemma 4). The case of *DL-Lite$_{horn}$* follows from Theorem 1. Finally, all other types of conservativity in this Lemma coincide with one of the considered above.

Robustness under vocabulary extensions is treated similarly.

C Decision procedures and complexity for *DL-Lite$_{bool}$*

Here we prove the complexity results for *DL-Lite$_{bool}$* stated in Theorems 1 and 6. To this end, recall the following result from [8]:

Theorem 7. *For DL-Lite$_{bool}$, the problem whether $\mathcal{T} \models C_1 \sqsubseteq C_2$ is* coNP*-complete.*

We use this result and Lemma 4 to prove

Theorem 8. *The deductive conservativity problem for DL-Lite$_{bool}$ TBoxes is Π_2^p-complete.*

Proof. Let $\mathcal{T}_1$, $\mathcal{T}_2$ be *DL-Lite$_{bool}$* TBoxes and Σ a signature. We may assume that $\Sigma \subseteq \mathsf{sig}(\mathcal{T}_1)$. We formulate a Σ_2^p algorithm deciding whether $\mathcal{T}_1 \cup \mathcal{T}_2$ is *not* a deductive conservative extension of $\mathcal{T}_1$ w.r.t. Σ.

1. Guess a $\Sigma Q_{\mathcal{T}_1 \cup \mathcal{T}_2}$-type $\boldsymbol{t}$. (Observe that the size of $\boldsymbol{t}$ is linear in the size of $\mathcal{T}_1 \cup \mathcal{T}_2$.)
2. Check, by calling an NP-oracle, whether (i) $\boldsymbol{t}$ is $\mathcal{T}_1$-consistent and whether (ii) $\boldsymbol{t}$ is not $\mathcal{T}_1 \cup \mathcal{T}_2$-consistent. Such an oracle exists by Theorem 7.
3. Return "$\mathcal{T}_1 \cup \mathcal{T}_2$ is not a deductive conservative extension of $\mathcal{T}_1$" if the answers to (i) and (ii) are both positive.

By Lemma 4, this algorithm is sound and complete.

We now come to the complexity result for *DL-Lite$_{bool}$* stated in Theorem 6. To prove this result we reduce precise satisfiability of a set of types (as stated in criterion **(tp)** of Lemma 5) to a satisfiability problem in propositional logic. Let Σ be a signature and Q a set of positive natural numbers containing 1. With every basic concept B of the form A or $\geq q\,R$ we associate a fresh *propositional variable* B^*, and, for a *DL-Lite$_{bool}$* concept C, denote by C^* the result of replacing each B in it with B^* (and $\sqcap$, $\sqcup$ with $\wedge$, $\vee$, respectively), for a ΣQ-type $\boldsymbol{t}$, denote by $\boldsymbol{t}^*$ the set $\{C^* \mid C \in \boldsymbol{t}\}$, and, for a TBox $\mathcal{T}$, denote by $\mathcal{T}^*$ the set $\{C_1^* \to C_2^* \mid C_1 \sqsubseteq C_2 \in \mathcal{T}\}$. Thus, C^* is a formula and $\boldsymbol{t}^*$, $\mathcal{T}^*$ are sets of formulas of *propositional logic*.

The following result follows immediately from [8]:

Lemma 15. *Let $\mathcal{T}$ be a DL-Lite$_{bool}$ TBox, $Q \supseteq Q_{\mathcal{T}}$, and Ω be a set of roles closed under inverse and containing all $\mathsf{sig}(\mathcal{T})$-roles. Then a set Ξ of ΣQ-types is precisely $\mathcal{T}$-realisable iff there is a set $\Omega_0 \subseteq \Omega$ closed under inverse such that:*

(t) *for each $\boldsymbol{t} \in \Xi$, $\boldsymbol{t}^* \cup \mathsf{Ax}(\mathcal{T}, \Omega_0)$ is satisfiable;*
(pw) *for each $R \in \Omega_0$, there is $\boldsymbol{t}_R \in \Xi$ such that $\boldsymbol{t}_R^* \cup \{(\geq 1\,R)^*\} \cup \mathsf{Ax}(\mathcal{T}, \Omega_0)$ is satisfiable,*

where

$$\mathsf{Ax}(\mathcal{T}, \Omega_0) = \mathcal{T}^* \quad \cup \quad \{\neg(\geq 1\,R)^* \mid R \in \Omega \setminus \Omega_0\} \quad \cup \quad \big\{(\geq q\,R)^* \to (\geq q'\,R)^* \mid R \in \Omega,\ q, q' \in Q,\ q > q'\big\}.$$

It follows, in particular, that, given $\mathcal{T}$ and a set Ξ of ΣQ-types, precise $\mathcal{T}$-realisability of Ξ is decidable in NP. To prove the complexity upper bound stated in Theorem 6, it will be sufficient to show that it is enough to consider sets Ξ of polynomial size in the size of $\mathcal{T}$.

Lemma 16. *Suppose that a set Ξ of $\Sigma Q_{\mathcal{T}_1 \cup \mathcal{T}_2}$-types is precisely $\mathcal{T}_1$-realisable but not precisely $\mathcal{T}_1 \cup \mathcal{T}_2$-realisable. Let Ω be the set of role names and their inverses that occur in $\mathcal{T}_1 \cup \mathcal{T}_2$. Then there is some $\Theta \subseteq \Xi$ with $|\Theta| \leq |\Omega| + 1$ such that Θ is precisely $\mathcal{T}_1$-realisable but not precisely $\mathcal{T}_1 \cup \mathcal{T}_2$-realisable.*

Proof. The proof follows from Lemmas 15. For every $\boldsymbol{t} \in \Xi$, there is $\Omega_0 \subseteq \Omega$ such that the set $\Theta_{\boldsymbol{t}} = \{\boldsymbol{t}\} \cup \{\boldsymbol{t}_R \mid R \in \Omega_0\}$ is precisely $\mathcal{T}_1$-realisable. But then at least one of these $\Theta_{\boldsymbol{t}}$, for $\boldsymbol{t} \in \Xi$, is as required, for otherwise, if all of them turn out to be precisely $\mathcal{T}_1 \cup \mathcal{T}_2$-realisable, the disjoint union of models $\mathcal{I}_{\boldsymbol{t}}$ for $\mathcal{T}_1 \cup \mathcal{T}_2$ precisely realising $\Theta_{\boldsymbol{t}}$ would precisely realise the whole Ξ, which is impossible.

Theorem 9. *The conservativity problems for DL-Lite$_{bool}$ mentioned in Theorem 6 are all Π_2^p-complete.*

Proof. We check criterion **(tp)** of Lemma 5. Let Σ be a signature and Ω the set of role names and their inverses that occur in $\mathcal{T}_1 \cup \mathcal{T}_2$. We may assume that $\Sigma \subseteq \mathsf{sig}(\mathcal{T}_1)$. By Lemma 15, for both $\mathcal{T} = \mathcal{T}_1$ and $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$, it is decidable in NP (in $|\mathcal{T}_1 \cup \mathcal{T}_1|$) whether a set Ξ of $\Sigma Q_{\mathcal{T}_1 \cup \mathcal{T}_2}$-types of size $\leq |\Omega| + 1$ is precisely $\mathcal{T}$-realisable. The Σ_2^p algorithm deciding whether there exists a set of $\Sigma Q_{\mathcal{T}_1 \cup \mathcal{T}_2}$-types that is precisely $\mathcal{T}_1$-realisable but not precisely $\mathcal{T}_1 \cup \mathcal{T}_2$-realisable is as follows:

1. Guess a set Ξ of $\Sigma Q_{\mathcal{T}_1 \cup \mathcal{T}_2}$-types of size $\leq |\Omega| + 1$.
2. Check, using an NP-oracle, whether (i) Ξ is precisely $\mathcal{T}_1$-realisable, and whether (ii) Ξ is not precisely $\mathcal{T}_1 \cup \mathcal{T}_2$-realisable.
3. Return "$\mathcal{T}_1 \cup \mathcal{T}_2$ is not a strong deductive conservative extension of $\mathcal{T}_1$" if the answers to (i) and (ii) are both positive.

By Lemmas 5 and 16, this algorithm is sound and complete.

D Decision procedures and complexity for *DL-Lite$_{horn}$*

The following is proved in [8]:

Theorem 10. *For DL-Lite$_{horn}$, the problem whether $\mathcal{T} \models C_1 \sqsubseteq C_2$ is* P-*complete.*

Theorem 11. *The deductive conservativity problem for DL-Lite$_{horn}$ TBoxes is* coNP-*complete.*

Proof. Observe that, by Lemmas 7 and 4, if $\mathcal{T}_1 \cup \mathcal{T}_2$ is not a deductive conservative extension of $\mathcal{T}_1$ in *DL-Lite$_{horn}$* w.r.t. Σ, then there exists a $\Sigma Q^h_{\mathcal{T}_1 \cup \mathcal{T}_2}$ concept inclusion witnessing this. For the NP-upper bound for non-conservativity, observe that such a witness concept inclusion is of polynomial size (in $|\mathcal{T}_1 \cup \mathcal{T}_2|$). Hence the algorithm non-deterministically guesses such a witness and then checks in polynomial time whether it is a consequence of $\mathcal{T}_1 \cup \mathcal{T}_2$ but not a consequence of $\mathcal{T}_1$. The coNP lower bound follows from the fact that conservativity is already coNP hard for the Horn fragment of propositional logic; see, e.g., [14].

The proof of NP-completeness of the query conservativity problem for *DL-Lite$_{horn}$* TBoxes is based on criterion **(tp)** of Lemma 10. Note that in precisely the same way as in the proof of Lemma 16 one can show now that if **(tp)** does not hold for some Ξ, then it does not hold for a Ξ with $|\Xi| \leq |\Omega| + 1$, where Ω is the set of all role names and their inverses that occur in $\mathcal{T}_1 \cup \mathcal{T}_2$.

In the formulations of the algorithms below we will be taking the *local closures* of types under the TBox rules. More precisely, given a type $\boldsymbol{t}$ and a *DL-Lite$_{horn}$* TBox $\mathcal{T}$, we denote by $\mathsf{cl}^\bullet_\mathcal{T}(\boldsymbol{t})$ the type where $\mathsf{cl}^\bullet_\mathcal{T}(\boldsymbol{t})^+$ is the result of applying iteratively the 'rules' from $\mathcal{T}$ to $\boldsymbol{t}^+$ in the Datalog manner: if $\bigsqcap B_i \sqsubseteq B \in \mathcal{T}$ and $B \notin \boldsymbol{t}^+$ then add B to $\boldsymbol{t}^+$. Note that the only difference between $\mathsf{cl}^\bullet_\mathcal{T}(\boldsymbol{t})$ and the 'global closure' $\mathsf{cl}_\mathcal{T}(\boldsymbol{t})$ is that the latter can contain $\bot$ even when $\bot \notin \mathsf{cl}^\bullet_\mathcal{T}(\boldsymbol{t})$. Indeed, consider the TBox

$$\mathcal{T} = \{A \sqsubseteq \exists R, \quad A \sqcap \exists R^- \sqsubseteq \bot, \quad \top \sqsubseteq A\}$$

and the type $\boldsymbol{t} = \{\top\}$. Then $\mathsf{cl}^\bullet_\mathcal{T}(\boldsymbol{t}) = \{\top, A, \exists R, \neg\exists R^-\}$, while $\mathsf{cl}_\mathcal{T}(\boldsymbol{t}) = \{\top, A, \exists R, \exists R^-, \bot\}$ because $\mathcal{T} \models \top \sqsubseteq \bot$; see Example 2.

Theorem 12. *The query conservativity problem for DL-Lite$_{horn}$ TBoxes is* coNP*-complete.*

Proof. We show a nondeterministic polynomial time algorithm for deciding whether $\mathcal{T}_1 \cup \mathcal{T}_2$ is *not* a query conservative extension of $\mathcal{T}_1$ in *DL-Lite$_{horn}$* w.r.t. Σ, where without loss of generality we may assume that $\Sigma \subseteq \mathsf{sig}(\mathcal{T}_1)$. The algorithm is formulated in a straightforward manner without a reduction to propositional logic. Let Ω be the set of all role names and their inverses that occur in $\mathcal{T}_1 \cup \mathcal{T}_2$.

1. Guess a set Ξ of $\Sigma Q_{\mathcal{T}_1 \cup \mathcal{T}_2}$-types of size $\leq |\Omega| + 1$.
2. For each $\boldsymbol{t} \in \Xi$, we compute (in time polynomial in $|\mathcal{T}_1 \cup \mathcal{T}_2|$) its local closure $\mathsf{cl}^{\bullet}_{\mathcal{T}_1}(\boldsymbol{t})$ and denote the set of all such closures by $\mathsf{cl}^{\bullet}_{\mathcal{T}_1}(\Xi)$ (these are all $\mathsf{sig}(\mathcal{T}_1) Q_{\mathcal{T}_1 \cup \mathcal{T}_2}$-types). Now, Ξ is precisely $\mathcal{T}_1$-realisable iff the following conditions hold:
   - $\boldsymbol{t} = \mathsf{cl}^{\bullet}_{\mathcal{T}_1}(\boldsymbol{t}) \restriction \Sigma Q_{\mathcal{T}_1 \cup \mathcal{T}_2}$ and $\bot \notin \boldsymbol{t}$, for all $\boldsymbol{t} \in \Xi$;
   - for every $\boldsymbol{t} \in \mathsf{cl}^{\bullet}_{\mathcal{T}_1}(\Xi)$ with $(\geq 1\, R) \in \boldsymbol{t}$, there is $\boldsymbol{t}' \in \mathsf{cl}^{\bullet}_{\mathcal{T}_1}(\Xi)$ with $(\geq 1\, \mathit{inv}(R)) \in \boldsymbol{t}'$.
3. If Ξ is precisely $\mathcal{T}_1$-realisable, then we do the following. First, compute $\Theta_0 = \mathsf{cl}^{\bullet}_{\mathcal{T}_1 \cup \mathcal{T}_2}(\Xi)$ and check whether
   - $\boldsymbol{t} = \mathsf{cl}^{\bullet}_{\mathcal{T}_1 \cup \mathcal{T}_1}(\boldsymbol{t}) \restriction \Sigma Q_{\mathcal{T}_1 \cup \mathcal{T}_2}$ and $\bot \notin \boldsymbol{t}$, for all $\boldsymbol{t} \in \Xi$.

   If this is not the case, stop with answer 'No.' Now, if Θ_i, $i \geq 0$, has already been computed and there is $\boldsymbol{t} \in \Theta_i$ with $(\geq 1\, R) \in \boldsymbol{t}$, for some role R, but there is no $\boldsymbol{t}' \in \Theta_i$ with $(\geq 1\, \mathit{inv}(R)) \in \boldsymbol{t}'$, then we construct the type $\boldsymbol{t}' = \mathsf{cl}^{\bullet}_{\mathcal{T}_1 \cup \mathcal{T}_2}(\{\geq 1\, \mathit{inv}(R)\})$, check whether the following holds
   - $\bot \notin \boldsymbol{t}'$ and there is $\boldsymbol{t} \in \Xi$ such that $\boldsymbol{t}'^{+} \restriction \Sigma Q_{\mathcal{T}_1 \cup \mathcal{T}_2} \subseteq \boldsymbol{t}^{+}$,

   and if it does, we add $\boldsymbol{t}'$ to Θ_i and denote the result by Θ_{i+1}; otherwise we terminate with answer 'No.' We stop when $\Theta_n = \Theta_{n+1}$. Clearly, all this can be done in polynomial time.

$\mathcal{T}_1 \cup \mathcal{T}_2$ is not a query conservative extension of $\mathcal{T}_1$ in *DL-Lite$_{horn}$* w.r.t. Σ iff there is a set Ξ guessed at step 1 such that the conditions at step 2 are satisfied, while step 3 terminates with answer 'No.'

Finally, we formulate a coNP algorithm for deciding strong deductive and query conservativity for *DL-Lite$_{horn}$* TBoxes, using criterion **(tp)** of Lemma 13. Note first that we again have the following:

Lemma 17. *Suppose that a set Ξ of $\Sigma Q_{\mathcal{T}_1 \cup \mathcal{T}_2}$-types is precisely $\mathcal{T}_1$-realisable but does not satisfy condition* **(tp)** *from Lemma 13. Let Ω be the set of role names and their inverses that occur in $\mathcal{T}_1 \cup \mathcal{T}_2$. Then there is some $\Theta \subseteq \Xi$ with $|\Theta| \leq |\Omega| + 1$ such that Θ is precisely $\mathcal{T}_1$-realisable but not join-precisely $\mathcal{T}_1 \cup \mathcal{T}_2$-realisable.*

Proof. As in the proof of Lemma 16, for every $\boldsymbol{t} \in \Xi$, we take $\Omega_0 \subseteq \Omega$ such that the set $\Theta_{\boldsymbol{t}} = \{\boldsymbol{t}\} \cup \{\boldsymbol{t}_R \mid R \in \Omega_0\}$ is precisely $\mathcal{T}_1$-realisable. We again claim that at least one of these $\Theta_{\boldsymbol{t}}$, for $\boldsymbol{t} \in \Xi$, is not join-precisely $\mathcal{T}_1 \cup \mathcal{T}_2$-realisable. Indeed, suppose, on the contrary, that **(tp)** holds for all the $\Theta_{\boldsymbol{t}}$. Then we have models $\mathcal{I}_{\boldsymbol{t}} \models \mathcal{T}_1 \cup \mathcal{T}_2$ such that $\mathcal{I}_{\boldsymbol{t}}$ realises all types in $\Theta_{\boldsymbol{t}}$ and each $\Sigma Q_{\mathcal{T}_1 \cup \mathcal{T}_2}$-type realised in $\mathcal{I}_{\boldsymbol{t}}$ is $\mathcal{T}_{\Theta_{\boldsymbol{t}}}$-consistent and h-contained in a type from $\Theta_{\boldsymbol{t}}$. Let $\mathcal{J}$ be the disjoint union of all these $\mathcal{I}_{\boldsymbol{t}}$. Clearly, $\mathcal{J} \models \mathcal{T}_1 \cup \mathcal{T}_2$, $\mathcal{J}$ realises all types in Ξ and each $\Sigma Q_{\mathcal{T}_1 \cup \mathcal{T}_2}$-type realised in $\mathcal{J}$ is h-contained in a type from Ξ. And since each $\Sigma Q_{\mathcal{T}_1 \cup \mathcal{T}_2}$-type realised in $\mathcal{J}$ is $\mathcal{T}_{\Theta_{\boldsymbol{t}}}$-consistent, it must be also $\mathcal{T}_{\Xi}$-consistent, as $\Theta_{\boldsymbol{t}} \subseteq \Xi$, and so $\mathcal{T}_{\Theta_{\boldsymbol{t}}} \supseteq \mathcal{T}_{\Xi}$. But then Ξ satisfies **(tp)**, which is a contradiction.

In the proof of the next theorem we will be taking the *local closures* of types under a *DL-Lite$_{horn}$* TBox $\mathcal{T}$ and the TBox $\mathcal{T}_{\Xi}$ induced by some set Ξ of types. More precisely, given a type $\boldsymbol{t}$, a *DL-Lite$_{horn}$* TBox $\mathcal{T}$ and a set Ξ of types, we denote by $\mathsf{cl}^{\bullet}_{\mathcal{T} \cup \mathcal{T}_{\Xi}}(\boldsymbol{t})$ the type where $\mathsf{cl}^{\bullet}_{\mathcal{T}}(\boldsymbol{t})^{+}$ is the result of the following iterative procedure:

1. if $\bigsqcap B_i \sqsubseteq B$ is in $\mathcal{T}$ and all conjuncts of $\bigsqcap B_i$ are in $\boldsymbol{t}^{+}$, then add B to $\boldsymbol{t}^{+}$;
2. if $\Lambda_{\boldsymbol{t}} = \{\boldsymbol{t}' \in \Xi \mid \boldsymbol{t}^{+} \subseteq \boldsymbol{t}'^{+}\}$ is empty then add $\bot$ to $\boldsymbol{t}^{+}$ and stop; otherwise add the concepts from $\bigcap_{\boldsymbol{t}' \in \Lambda_{\boldsymbol{t}}} \boldsymbol{t}'^{+}$ to $\boldsymbol{t}^{+}$;
3. stop if $\boldsymbol{t}^{+}$ has not been incremented, go to step 1 otherwise.

Theorem 13. *The strong deductive and query conservativity problems for DL-Lite$_{horn}$ TBoxes are* coNP*-complete.*

Proof. We show a nondeterministic polynomial time algorithm for deciding whether $\mathcal{T}_1 \cup \mathcal{T}_2$ is *not* a strong deductive (or query) conservative extension of $\mathcal{T}_1$ in *DL-Lite$_{horn}$* w.r.t. Σ, where without loss of generality we again assume that $\Sigma \subseteq \mathsf{sig}(\mathcal{T}_1)$. Let Ω be the set of all role names and their inverses that occur in $\mathcal{T}_1 \cup \mathcal{T}_2$.

1. Guess a set Ξ of $\Sigma Q_{\mathcal{T}_1 \cup \mathcal{T}_2}$-types of size $\leq |\Omega| + 1$.
2. For each $\boldsymbol{t} \in \Xi$, we compute (in time polynomial in $|\mathcal{T}_1 \cup \mathcal{T}_2|$) its $\mathcal{T}_1$-closure $\mathsf{cl}^{\bullet}_{\mathcal{T}_1}(\boldsymbol{t})$ and denote the set of all such closures by $\mathsf{cl}^{\bullet}_{\mathcal{T}_1}(\Xi)$ (these are all $\mathsf{sig}(\mathcal{T}_1) Q_{\mathcal{T}_1 \cup \mathcal{T}_2}$-types). Now, Ξ is precisely $\mathcal{T}_1$-realisable iff the following conditions hold:
   - $\boldsymbol{t} = \mathsf{cl}^{\bullet}_{\mathcal{T}_1}(\boldsymbol{t}) \restriction \Sigma Q_{\mathcal{T}_1 \cup \mathcal{T}_2}$ and $\bot \notin \boldsymbol{t}$, for all $\boldsymbol{t} \in \Xi$;
   - for every $\boldsymbol{t} \in \mathsf{cl}^{\bullet}_{\mathcal{T}_1}(\Xi)$ with $(\geq 1\, R) \in \boldsymbol{t}$, there is $\boldsymbol{t}' \in \mathsf{cl}^{\bullet}_{\mathcal{T}_1}(\Xi)$ with $(\geq 1\, inv(R)) \in \boldsymbol{t}'$.
3. If Ξ is precisely $\mathcal{T}_1$-realisable, then we do the following. First, compute $\Theta_0 = \mathsf{cl}^{\bullet}_{\mathcal{T}_1 \cup \mathcal{T}_2}(\Xi)$ and check whether
   - $\boldsymbol{t} = \mathsf{cl}^{\bullet}_{\mathcal{T}_1 \cup \mathcal{T}_1}(\boldsymbol{t}) \restriction \Sigma Q_{\mathcal{T}_1 \cup \mathcal{T}_2}$ and $\bot \notin \boldsymbol{t}$, for all $\boldsymbol{t} \in \Xi$.

   If this is not the case, stop with answer 'No.' Now, if Θ_i, $i \geq 0$, has already been computed and there is $\boldsymbol{t} \in \Theta_i$ with $(\geq 1\, R) \in \boldsymbol{t}$, for some role R, but there is no $\boldsymbol{t}' \in \Theta_i$ with $(\geq 1\, inv(R)) \in \boldsymbol{t}'$, then we construct the type $\boldsymbol{t}' = \mathsf{cl}^{\bullet}_{\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_{\Xi}}(\{\geq 1\, inv(R)\})$ according to the procedure above, check whether the following holds
   - $\bot \notin \boldsymbol{t}'$ and there is $\boldsymbol{t} \in \Xi$ such that $\boldsymbol{t}'^{+} \restriction \Sigma Q_{\mathcal{T}_1 \cup \mathcal{T}_2} \subseteq \boldsymbol{t}^{+}$,

   and if it does, we add $\boldsymbol{t}'$ to Θ_i and denote the result by Θ_{i+1}; otherwise we terminate with answer 'No.' We stop when $\Theta_n = \Theta_{n+1}$. Clearly, all this can be done in polynomial time.

$\mathcal{T}_1 \cup \mathcal{T}_2$ is not a strong deductive (or query) conservative extension of $\mathcal{T}_1$ in *DL-Lite$_{horn}$* w.r.t. Σ iff there is a set Ξ guessed at step 1 such that the conditions at step 2 are satisfied, while step 3 terminates with answer 'No.'