Complexity of Hybrid Logics over Transitive Frames

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Complexity of Hybrid Logics over Transitive Frames

- Modal Propositional Logic
- Temporal Logic
- Why Transitive Frames?
- Hybrid Logic
- Overview and Open Questions

Modal Propositional Logic

Modal Logic

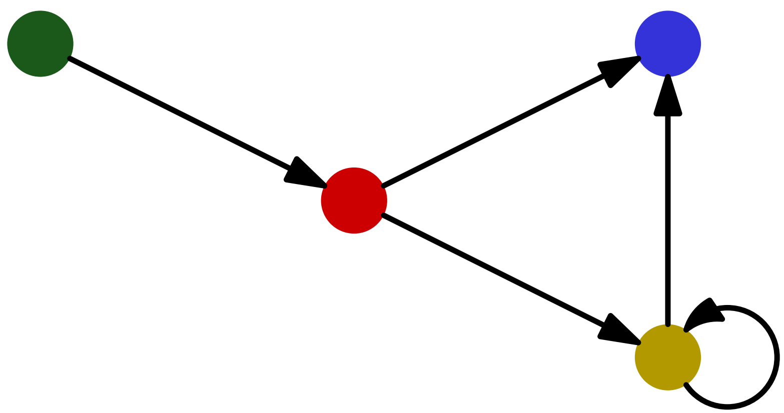
Syntax

- Formulas: $\varphi ::= p \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \Diamond\varphi$,
  where p is an atomic proposition
- Abbreviations $\vee$, $\longrightarrow$, $\leftrightarrow$ as usual; $\Box\varphi = \neg\Diamond\neg\varphi$
- Language: ML

Modal Logic

Semantics

- Models $\mathfrak{M} = (W, R, V)$
- Frames $\mathfrak{F} = (W, R)$



arbitrary frame

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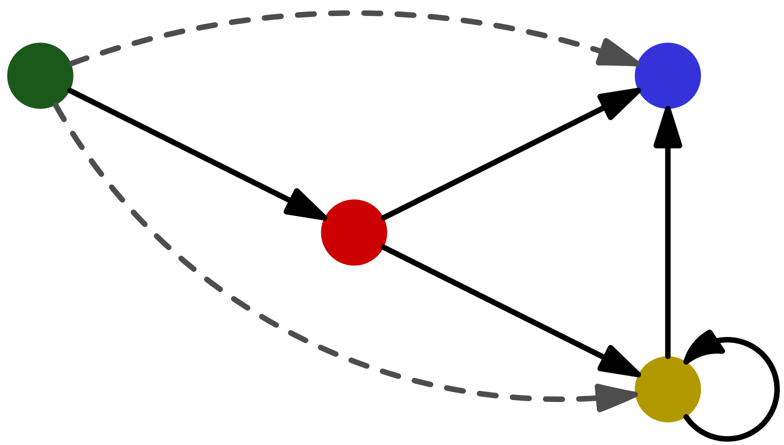

transitive frame

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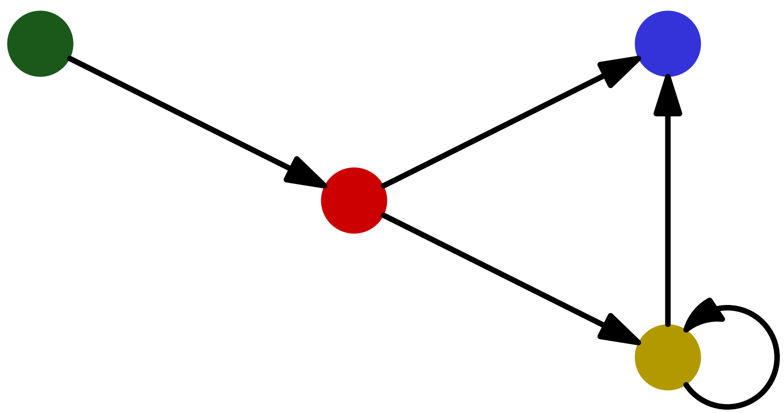

transitive frame

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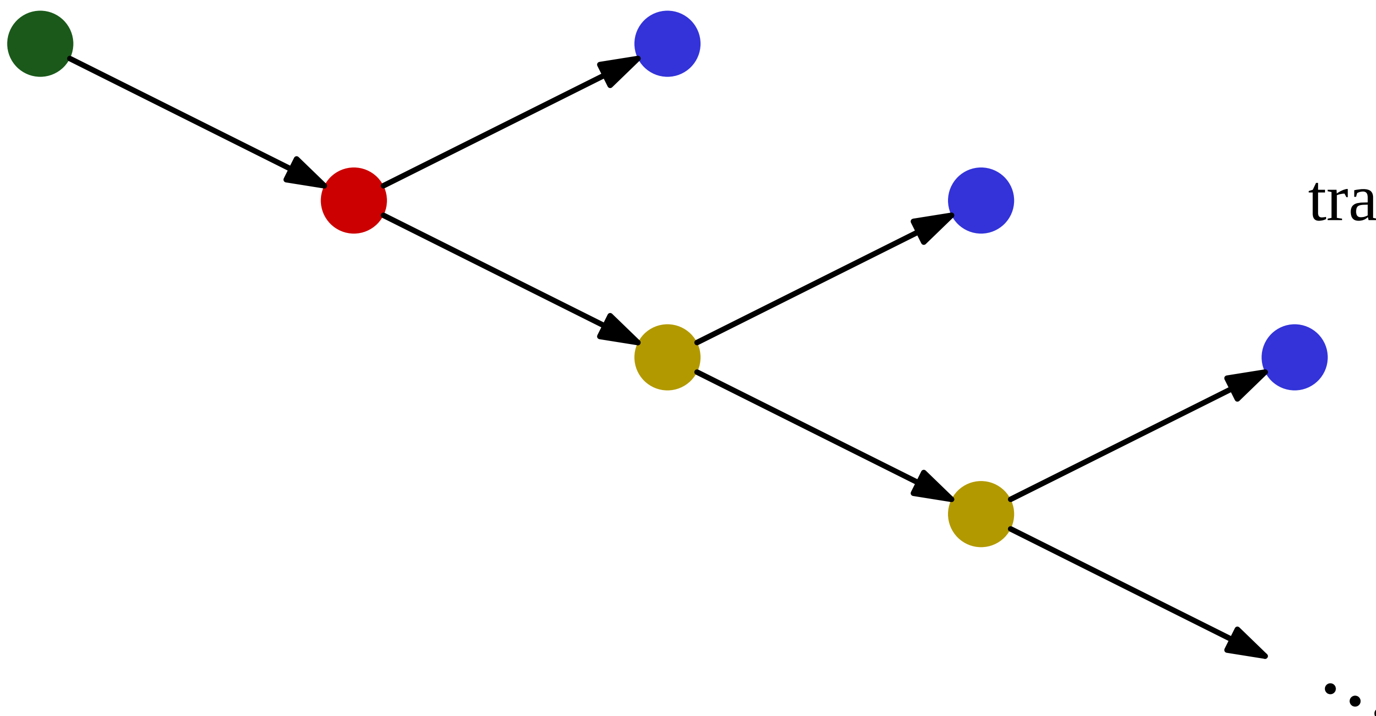

transitive tree

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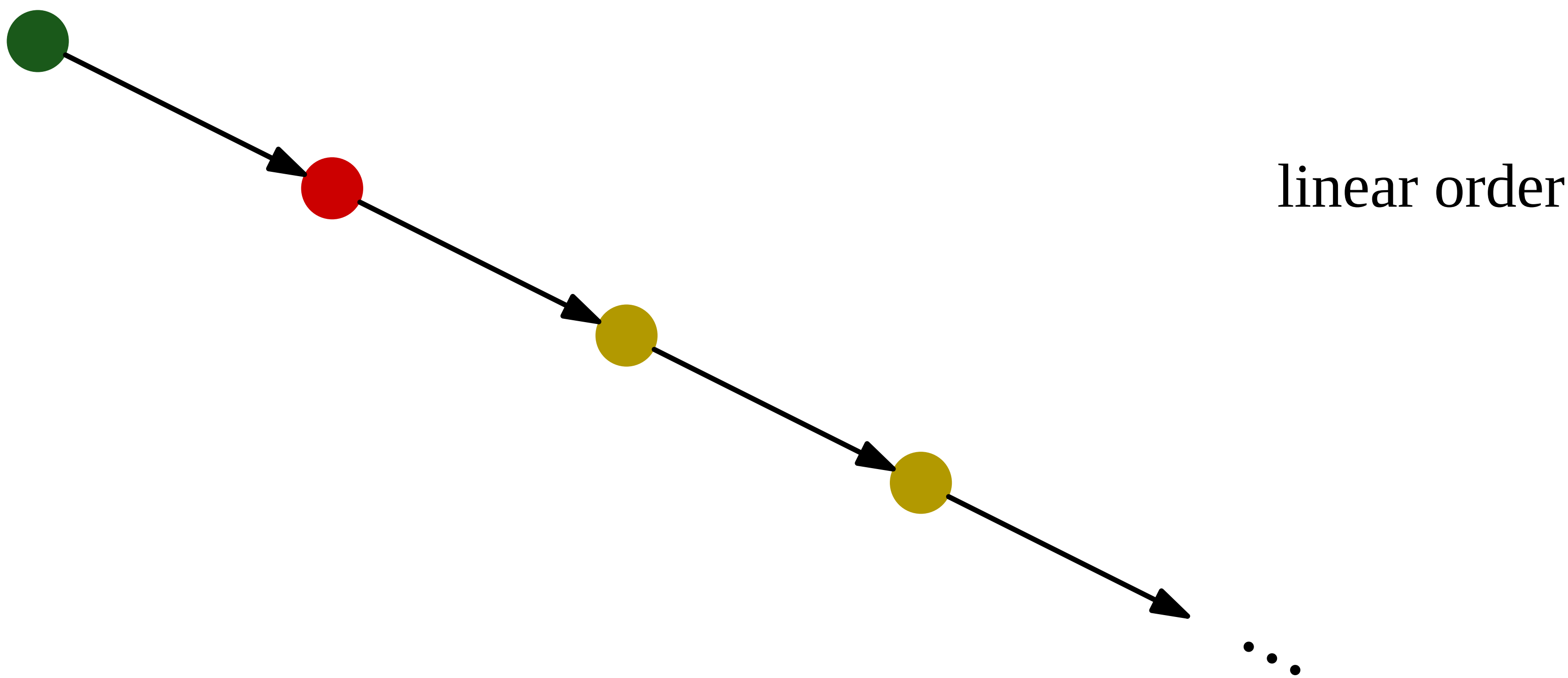

Modal Logic

Truth and Satisfiability

- Truth is defined as usual.
- We consider the satisfiability problem ML-SAT:
  Given a formula φ,
  is there a model $\mathfrak{M} = (W, R, V)$ and a point $w \in W$,
  such that $\mathfrak{M}, w \models \varphi$?

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  is there a model $\mathfrak{M} = (W, R, V)$ and a point $w \in W$,
  such that $\mathfrak{M}, w \models \varphi$?
- ML-SAT is PSPACE-complete. [LADNER 1977]
- Under restricted frame classes:
  - PSPACE-complete over transitive or reflexive frames
  - NP-complete over equivalence relations

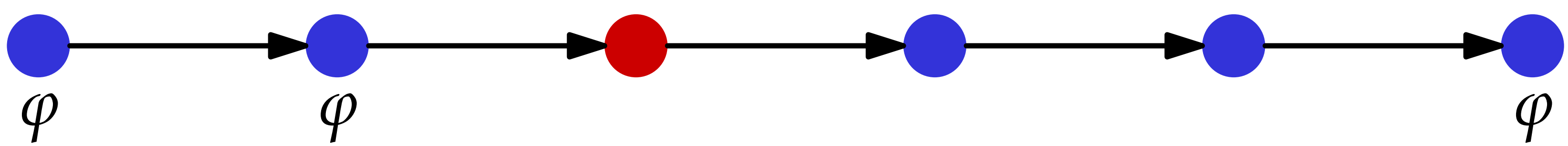
[LADNER 1977]

Temporal Logic

Temporal Logic

Basic Temporal Operators

- F, G ("*Future*", "*Going to*") — other names for $\Diamond, \Box$
- P, H ("*Past*", "*Has been*") — correspond to $\Diamond^{-}, \Box^{-}$
- Example:



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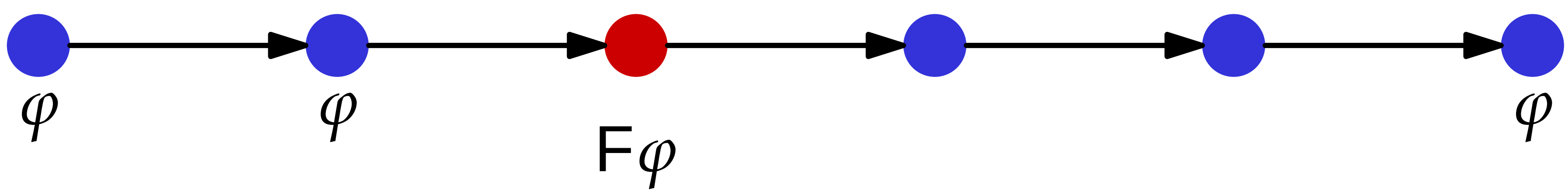

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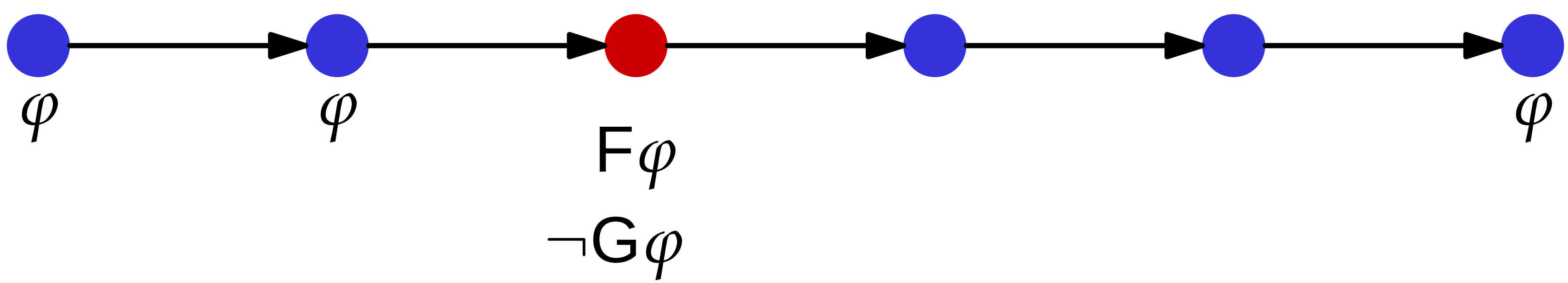

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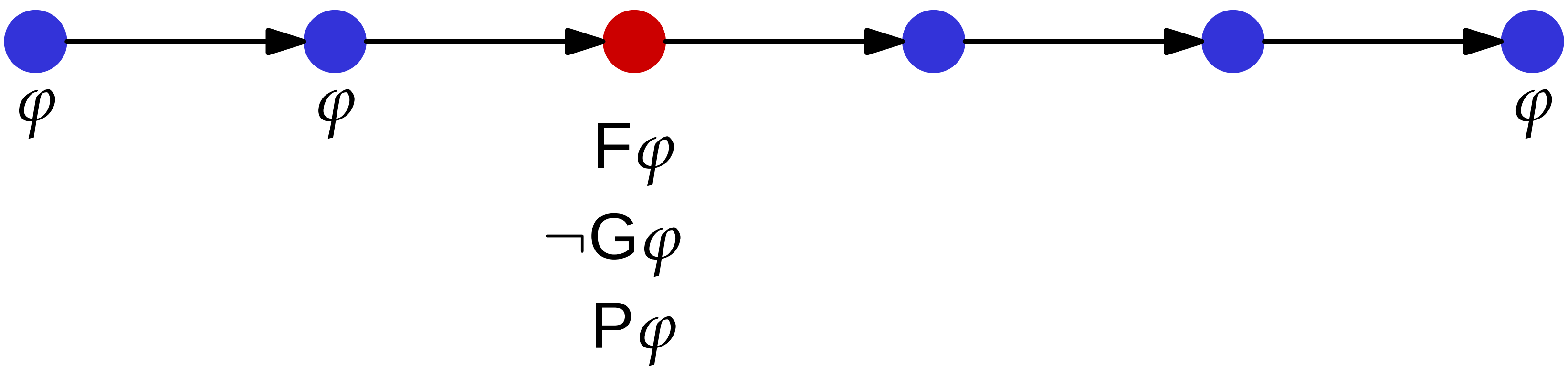

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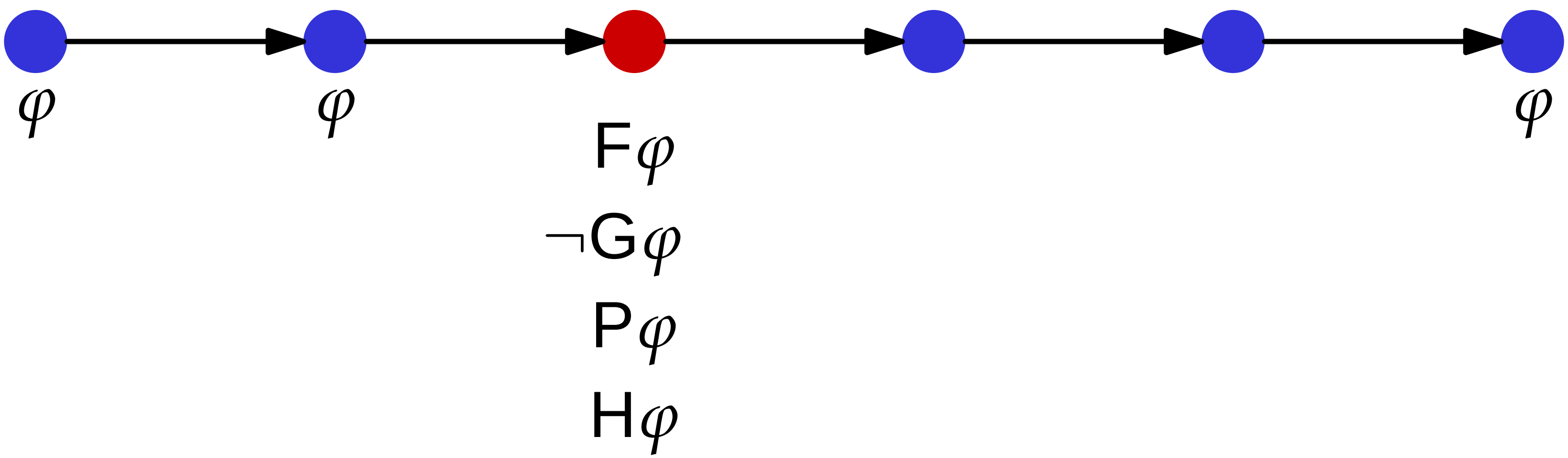

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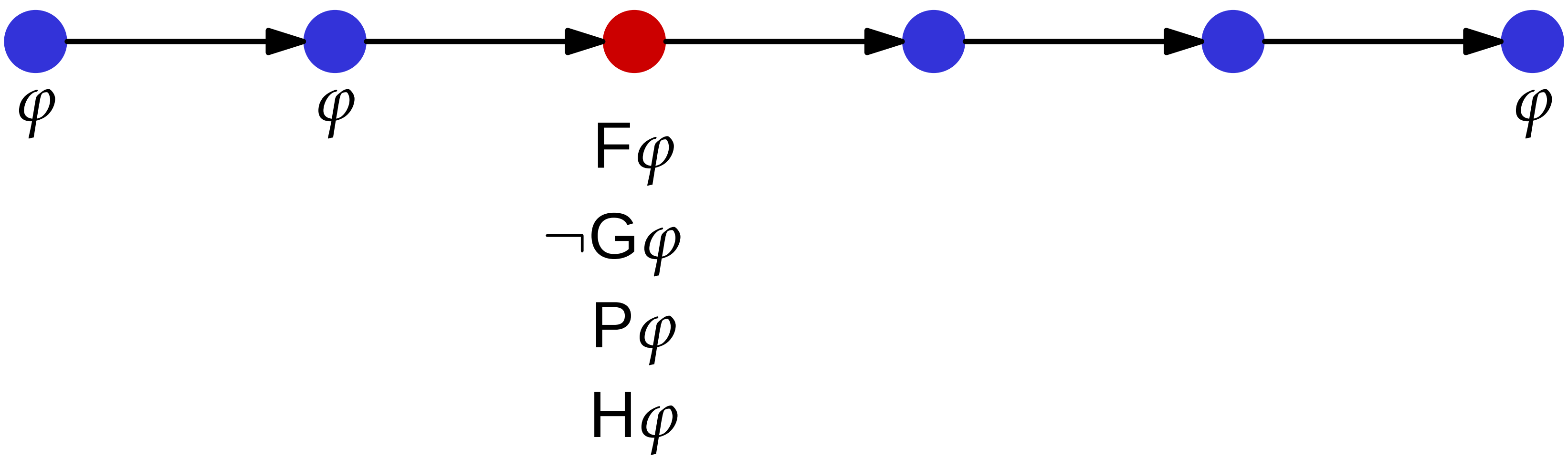
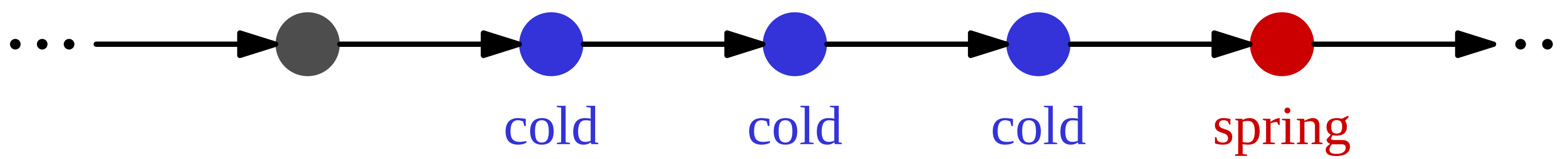


- $\mathrm{ML}_{\mathsf{F,P}}$-SAT remains PSPACE-complete. [SPAAN 1993]

Temporal Logic

Until and Since

- “There will be a point in the future, at which it will *be spring*, and from now until then it will always *be cold*.”



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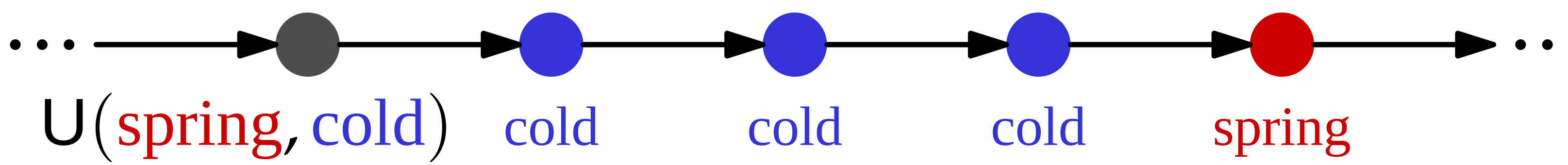

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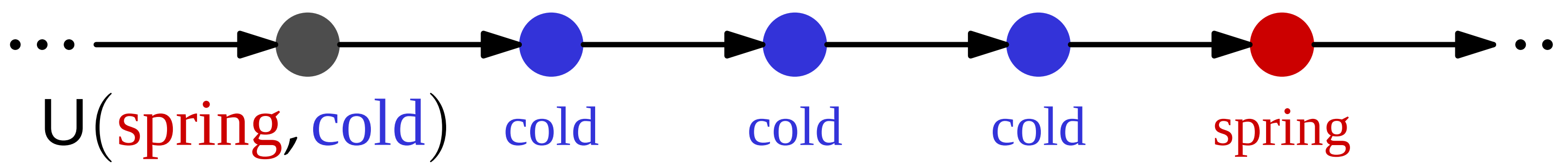

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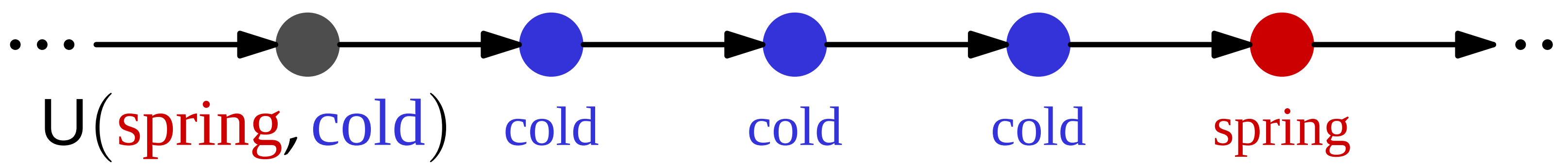

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- $\mathrm{ML}_{\mathsf{U},\mathsf{S}}$-SAT over linear orders: PSPACE-complete.
  (ML-SAT over linear orders: NP-complete.)
  [Sistla, Clarke 1985 / Ono, Nakamura 1980]

Why Transitive Frames?

Why Transitive Frames?

- Transitivity is a property most temporal applications have in common.
- Can we exactly locate the decrease in complexity taking place when proceeding from arbitrary frames to linear orders?

Logic	arbitrary frames	...	linear orders
ML	PSPACE	...	NP
P	PSPACE	...	NP
$i, @, \mathsf{P}$	EXP	...	NP
$i, \downarrow$	coRE	...	NP

Hybrid Logic I

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Nominals

- Allow for explicit naming of points.
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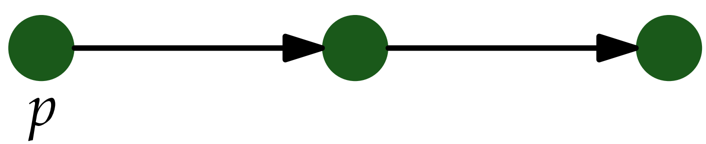

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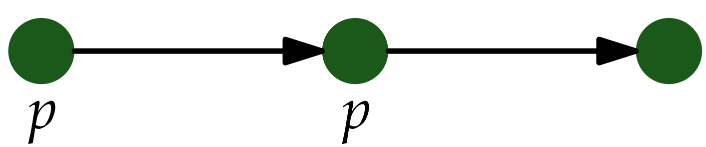

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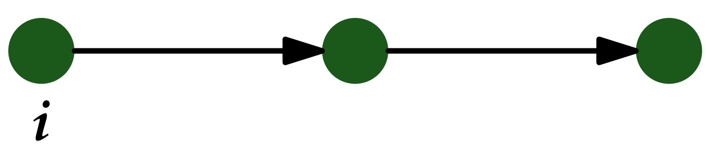

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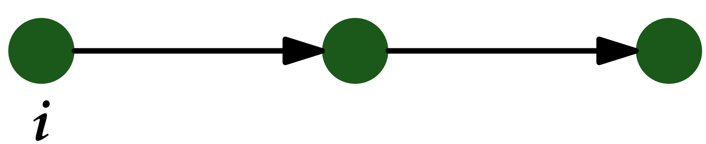

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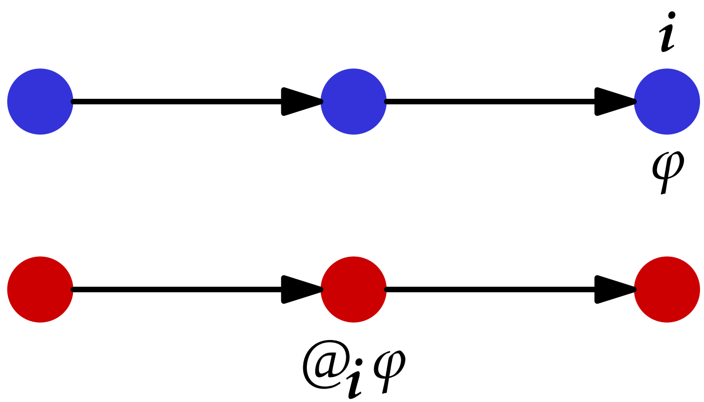
- HL = ML "plus" nominals.

Hybrid Logic I

The @ Operator

- "Jumps" to named points.
- $\mathfrak{M}, w \models @_{\boldsymbol{i}}\varphi \quad \text{iff} \quad \mathfrak{M}, V(\boldsymbol{i}) \models \varphi$
- Example:



Complexity of satisfiability?

Hybrid Logic I

$HL^{@}$-SAT

Over arbitrary and transitive frames: PSPACE-complete.

[ARECES, BLACKBURN, MARX 1999/2000]

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$HL^{@}_{F,P}$-SAT

Over arbitrary and transitive frames: EXPTIME-complete. [ABM]

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- Over arbitrary frames: EXPTIME-complete. [ABM]

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  - EXPTIME-hard and in 2EXPTIME. [MSSW 2005]
  - Lower bound holds for ML_U-SAT.

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- Over transitive trees:
  - EXPTIME-complete. [MSSW 2005]
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Hybrid Logic II

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The $\downarrow$ Operator

- $\downarrow x.\varphi$: Name the current point x and evaluate φ, treating all occurrences of x in φ as nominals for this point.

Hybrid Logic II

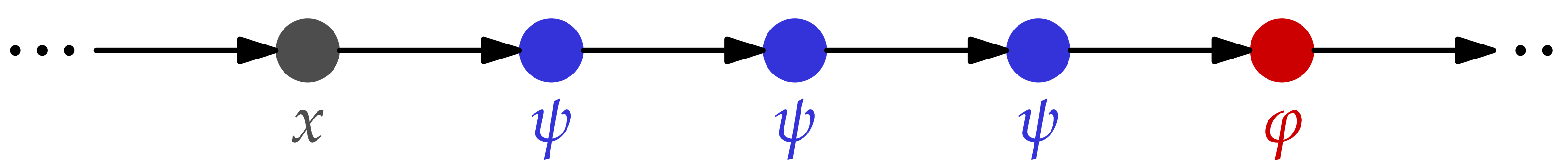
The ↓ Operator

- $\downarrow x.\varphi$: Name the current point x and evaluate φ, treating all occurrences of x in φ as nominals for this point.
- Example: U can be expressed by means of ↓ *and* @:

$$\mathsf{U}(\varphi,\psi) \equiv \downarrow x.\Diamond\downarrow y.\varphi \wedge @_x\Box(\Diamond y \longrightarrow \psi)$$

- or, alternatively, by means of ↓ *and* past modalities:

$$\mathsf{U}(\varphi,\psi) \equiv \downarrow x.\mathsf{F}\big(\varphi \wedge \mathsf{H}(\mathsf{P}x \longrightarrow \psi)\big)$$



Hybrid Logic II

Satisfiability for ↓ languages

- Over arbitrary frames, $HL^{\downarrow}$ is undecidable.
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  - $HL^{\downarrow,@}$ and $HL^{\downarrow}_{F,P}$ are undecidable. [MSSW 2005]

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Satisfiability for $\downarrow$ languages

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- Over transitive frames:
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  - $\mathrm{HL}^{\downarrow,@}$ and $\mathrm{HL}^{\downarrow}_{\mathsf{F},\mathsf{P}}$ are undecidable. [MSSW 2005]
- Over transitive trees:
  - $\downarrow$ *alone* is useless.
  - $\mathrm{HL}^{\downarrow,@}$ and $\mathrm{HL}^{\downarrow}_{\mathsf{F},\mathsf{P}}$ are nonelementarily decidable. [MSSW 2005]

$$\left(\mathrm{ELEMENTARY} = \bigcup \mathrm{DTIME}\left(2^{2^{\cdot^{\cdot^{\cdot^{2^n}}}}}\right)\right)$$

Overview and Open Questions

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$i, @, \mathsf{P}$	EXP	EXP	PSPACE	NP
$i, @, \mathsf{U}, \mathsf{S}$	EXP	**in 2EXP, EXP-hard**	**EXP**	PSPACE-hard
$i, \downarrow$	coRE	**NEXP**	PSPACE	NP
$i, \downarrow, @$	coRE	**coRE**	**nonel.**	*nonel.*
$i, \downarrow, \mathsf{P}$	coRE	**coRE**	**nonel.**	nonel.
$i, \downarrow, @, \mathsf{P}$	coRE	**coRE**	**nonel.**	nonel.

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$i, \downarrow, @, \mathsf{P}$	coRE	**coRE**	**nonel.**	nonel.

Thank you!