# Update Using Subset Logic

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**Abstract**

We introduce a new logic by modifying Moss-Parikh subset logic that addresses change during knowledge acquisition. This notion of change does not necessarily result in a restriction of possible alternatives, but rather tracks change on each of them. We prove completeness and decidability.

## 1 Introduction

Subset logic is a bimodal logic that combines two modal operators, one corresponding to knowledge, and one corresponding to effort, and models the increase of knowledge after a larger amount of resources has been spent in acquiring it. Subset logic has been introduced by Moss and Parikh who also established the basic results ([MP92],[DMP96]). A great deal of further research has been devoted to characterizing the underlying structure of subsets using axioms of this logic. For example, the system topologic has been found complete with respect to topological spaces ([Geo93]). Variants of this logic have also been developed to address knowledge after program termination and time passing ([Hei99],[Hei07]).

The main novelty of subset logic is its semantics, where, after fixing a space of subsets of a set of worlds, sentences are interpreted over a pair $(x,U)$, where $x$ is the actual world the agent resides in, and $U$ is the view the agent has. The agent's view consists of those worlds the agent considers possible. We can represent effort by restricting the agent's view. Restricting the view means that the agent cancels out some of the alternatives, and, as a result, increase of knowledge occurs. In this paper, we would like to explore the possibility of using subsets of world to model any kind of change, rather than change that corresponds to increase of knowledge. Consider the following example.

**Example 1** Imagine a room with a table and two cubes: a red one and a blue one. The agent is outside the room but knows that one of the cubes is on the table and one is on the floor. Now, suppose that the agent instructs a robot to enter the room and place the red cube on the floor. The update of the agent's knowledge base after the robot's action can be modeled as follows. Denote the sentence "the red cube is on the floor" with $r$, and similarly with $b$ for the blue cube. The initial view is $U = \{w_1, w_2\}$ where $w_1 = \{r \wedge \neg b\}$ and $w_2 = \{\neg r \wedge b\}$. After the robot is instructed to place the red cube on the floor the first possibility persists while the second possibility turns into $w_2' = \{r \wedge b\}$. The resulting view of the agent is $U' = \{w_1, w_2'\}$.

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Note that, in the above example, $U'$ is not a subset of $U$ and change does not result in an increase but rather in an update of knowledge (in the sense of [KM91]). In particular, the sentence

$$\mathsf{K}(r \leftrightarrow \neg b)$$

is true at $U$ but false in $U'$. Further, the resulting subset is determined by the transformation of its components. Before instructing the robot we do not know whether the red cube is on the floor:

$$\neg\mathsf{K}\neg r$$

but after instructing the robot (call this action $a$) the following holds:

$$[a]\mathsf{K}r.$$

Our modification is threefold:

- We consider subsets that do not necessarily form a topological space but rather they are determined by the accessibility relations. This is not contrary to the basic intuition behind the Moss-Parikh logics but rather complementary. We do not want to express the structure of subsets but rather the structure of actions that restrict the agent's view to those subsets.
- We consider changes that do not necessarily result in a smaller subset. Frequently, we reason with defeasible knowledge or we jump to conclusions as in nonmonotonic reasoning. Other times, we need to revise our beliefs. In such cases, the resulting epistemic state is not a refinement but rather a transformation of the original one.
- We make explicit the accessibility relations that bring about different forms of change. Such actions can be the result of a program, a game move, pieces of information about a changing world, or, simply, the passage of time.

We view this logic as a tool for studying the transformations of knowledge in a more general setting much like dynamic epistemic logic ([DvdHK07]), although we restrict our attention to a single agent (for a multiple agent approach using subset logic see [Hei08]). Systems that combine two or more modal logics are nothing new ([KW91],[GS98]). Their theory, in the simple cases, is straightforward. Similarly, basic results such as completeness for Kripke models and decidability are straightforward once those have been obtained for the individual logics separately. However, our completeness and decidability results refer *not* to the usual frame models with accessibility relations but, instead, we use subsets.

In the next section, we define the subset logic **SC** for reasoning about change and prove a normal form theorem. Then, we apply the normal form theorem to prove completeness in section 3 and decidability in section 4. We conclude with a sketch of a logic that incorporates actions.

# 2 Syntax and Semantics

We follow the notation of [MP92].

Our language is bimodal and propositional. We start with a countable set $\mathsf{Atom}$ of *atomic formulas* containing two distinguished elements $\top$ and $\bot$. Then the *language* $\mathscr{L}$ is the least set such that $\mathsf{Atom} \subseteq \mathscr{L}$ and closed under the following rules:

$$\frac{\phi, \psi \in \mathscr{L}}{\phi \wedge \psi \in \mathscr{L}} \qquad \frac{\phi \in \mathscr{L}}{\neg\phi, \Box\phi, \mathsf{K}\phi \in \mathscr{L}}$$

The above language can be interpreted using subsets as follows:

**Definition 2** Let $X$ be a set, $R$ a binary relation on $X$, i.e., $R \subseteq X \times X$ called *accessibility*, and $\mathscr{O}$ a subset of the powerset of $X$, i.e. $\mathscr{O} \subseteq \mathscr{P}(X)$ such that $X \in \mathscr{O}$. We denote the set $\{(x,U) : x \in X, U \in \mathscr{O}, \text{ and } x \in U\} \subseteq X \times \mathscr{O}$ by $X \dot{\times} \mathscr{O}$. For each $U \in \mathscr{O}$, let $U^R$ be the set of the elements accessible from $U$, that is, the set $\{y : (x,y) \in R, x \in U\}$. The set $\mathscr{O}$ will be called *R-closed* if whenever $U \in \mathscr{O}$ then $U^R \in \mathscr{O}$.

Let $\mathscr{O}$ be $R$-closed, then the triple $\langle X, R, \mathscr{O}\rangle$ will be called a *subset frame*. A *model* is a quadruple $\langle X, R, \mathscr{O}, i\rangle$, where $\langle X, R, \mathscr{O}\rangle$ is a subset frame and $i$ a map from $\mathsf{Atom}$ to $\mathscr{P}(X)$ with $i(\top) = X$ and $i(\bot) = \emptyset$ called *initial interpretation*.

**Definition 3** The *satisfaction relation* $\models_{\mathscr{M}}$, where $\mathscr{M}$ is the model $\langle X, \mathscr{O}, R, i\rangle$, is a subset of $(X \dot{\times} \mathscr{O}) \times \mathscr{L}$ defined recursively by (we write $x, U \models_{\mathscr{M}} \phi$ instead of $((x,U),\phi) \in \models_{\mathscr{M}}$):

$$
\begin{array}{ll}
x,U \models_{\mathscr{M}} A & \text{iff } x \in i(A), \text{ where } A \in \mathsf{Atom} \\
x,U \models_{\mathscr{M}} \phi \wedge \psi & \text{iff } x,U \models_{\mathscr{M}} \phi \text{ and } x,U \models_{\mathscr{M}} \psi \\
x,U \models_{\mathscr{M}} \neg\phi & \text{iff } x,U \not\models_{\mathscr{M}} \phi \\
x,U \models_{\mathscr{M}} \mathsf{K}\phi & \text{iff for all } y \in U,\ y,U \models_{\mathscr{M}} \phi \\
x,U \models_{\mathscr{M}} \Box\phi & \text{iff for all } y \in X \text{ such that } (x,y) \in R,\ y,U^R \models_{\mathscr{M}} \phi.
\end{array}
$$

If $x,U \models_{\mathscr{M}} \phi$ for all $(x,U)$ belonging to $X \dot{\times} \mathscr{O}$ then $\phi$ is *valid* in $\mathscr{M}$, denoted by $\mathscr{M} \models \phi$.

We abbreviate $\neg\Box\neg\phi$ and $\neg\mathsf{K}\neg\phi$ by $\Diamond\phi$ and $\mathsf{L}\phi$ respectively. We have that

$$
\begin{array}{ll}
x,U \models_{\mathscr{M}} \mathsf{L}\phi & \text{if there exists } y \in U \text{ such that } y,U \models_{\mathscr{M}} \phi \\
x,U \models_{\mathscr{M}} \Diamond\phi & \text{if there exists } y \in X \text{ such that } (x,y) \in R \text{ and } y,U^R \models_{\mathscr{M}} \phi.
\end{array}
$$

The axiom system **SC** consists of axiom schemes 1 through 8 and rules of table 1 (see page 4). In other words, we require that $\Box$ satisfies the **K** (normality) axiom and $\mathsf{K}$ satisfies the **S5** axioms. We have two interaction axioms. Axiom 7 is akin to "perfect recall" of [ST08] for a single modality. Axiom 8 identifies $\Box$ and $\Diamond$ for knowledge sentences and axiomatizes the functionality of accessibility relation on subsets. Observe that the following basic axiom of subset logic is not valid in **SC**:

$$(A \to \Box A) \wedge (\neg A \to \Box\neg A), \text{ for } A \in \mathsf{Atom}.$$

The following holds:

**Theorem 4** *The axioms and rules of* **SC** *are sound with respect to subset frames.*

We will prove that the logic **SC** has a normal form (Theorem 8). Through this normal form we can characterize accessibility by decomposing the theories of the canonical model (Proposition 16). This approach is motivated by [Geo93].

**Definition 5** Let $\mathscr{L}^{\Box} \subseteq \mathscr{L}$ be the set of formulas generated by the following rules:

$$\mathsf{Atom} \subseteq \mathscr{L}^{\Box} \qquad \frac{\phi, \psi \in \mathscr{L}^{\Box}}{\phi \wedge \psi \in \mathscr{L}^{\Box}} \qquad \frac{\phi \in \mathscr{L}^{\Box}}{\neg\phi, \Box\phi \in \mathscr{L}^{\Box}}$$

Let $\mathscr{L}^{\mathsf{K}}$ be the set $\{\mathsf{K}\phi, \mathsf{L}\phi | \phi \in \mathscr{L}^{\Box}\}$.

**Lemma 6** *The following are theorems of* **SC**.

1. $\Diamond(\phi \wedge \mathsf{K}\psi) \leftrightarrow \Diamond\phi \wedge \Diamond\mathsf{K}\psi$.
2. $\Diamond\top \wedge \Box\mathsf{K}\phi \to \mathsf{K}\Box\phi$
3. $\Diamond\mathsf{L}\phi \to \mathsf{L}\Diamond\phi$
4. $\Diamond\top \wedge \Box\mathsf{L}\phi \to \Diamond\mathsf{L}\phi$
5. $\mathsf{L}\Diamond\phi \to \Box\mathsf{L}\phi$

*Proof.*

For 1, the one implication is straightforward. For the other

1. $\Diamond\phi \wedge \Diamond\mathsf{K}\psi \to \Diamond\phi \wedge \Box\mathsf{K}\psi$ by Axioms 7 and 8
2. $\Diamond\phi \wedge \Box\mathsf{K}\psi \to \Diamond(\phi \wedge \mathsf{K}\psi)$ in a normal system.

2 follows from Axiom 8. The rest are contrapositives of Axiom 7, Case 2, and Axiom 8.

∎

**Axioms**

1. All propositional tautologies
2. $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$
3. $\mathsf{K}(\phi \to \psi) \to (\mathsf{K}\phi \to \mathsf{K}\psi)$
4. $\mathsf{K}\phi \to \phi$
5. $\mathsf{K}\phi \to \mathsf{K}\mathsf{K}\phi$
6. $\phi \to \mathsf{K}\mathsf{L}\phi$
7. $\mathsf{K}\Box\phi \to \Box\mathsf{K}\phi$
8. $\Diamond\mathsf{K}\phi \to \mathsf{K}\Box\phi$

**Rules**

$$\frac{\phi \to \psi, \phi}{\psi}\ \text{SC}$$

$$\frac{\phi}{\mathsf{K}\phi}\ \mathsf{K}\text{-Necessitation} \qquad \frac{\phi}{\Box\phi}\ \Box\text{-Necessitation}$$

Table 1: Axioms and Rules of **SC**

**Definition 7** 1. $\phi$ is in *prime normal form* (PNF) if it has the form

$$\psi \wedge \mathsf{K}\psi' \wedge \bigwedge_{i=1}^{n} \mathsf{L}\psi_i$$

where $\psi, \psi', \psi_i \in \mathscr{L}^{\Box}$ and $n$ is finite.

2. $\phi$ is in *disjunctive normal form* (DNF) if it has the form $\bigvee_{i=1}^{m} \phi_i$, where each $\phi_i$ is in PNF and $m$ is finite.

We shall omit the cardinality of (finite) conjunctions and disjunctions, writing, e.g., $\bigvee_i \phi_i$ instead of $\bigvee_{i=1}^{n} \phi_i$. Suppose that $\phi$ is a formula in the following form

$$\bigwedge_i \left( \psi_i \vee \mathsf{L}\psi_i' \vee \bigvee_j \mathsf{K}\psi_i^j \right),$$

where $\psi_i, \psi_i', \psi_i^j \in \mathscr{L}^{\Box}$. We shall call such a form *conjunctive normal form* (CNF). Using the distributive laws, we may show that DNF and CNF are effectively interchangeable up to equivalence.

**Theorem 8 (DNF)** *For every $\phi \in \mathscr{L}$, there is (effectively) a $\psi$ in DNF such that*

$$\vdash_{\mathbf{SC}} \phi \equiv \psi.$$

*Proof.* By induction on the logical structure of $\phi$.

- If $\phi = A$, where $A$ is atomic, the result is immediate because the set of atomic formulas belongs to $\mathscr{L}^{\Box}$ and $A$ is in PNF.
- Suppose that $\phi = \neg\psi$. Then, by induction hypothesis, $\psi$ is equivalent to a formula in DNF, which implies that $\phi$ is equivalent to a formula in CNF and, by the above discussion, is equivalent to a formula in DNF.
- If $\phi = \psi \vee \chi$ then $\phi$ is equivalent to a disjunction of two formulas in DNF, i.e. is itself in DNF.
- If $\phi = \mathsf{K}\psi$ then $\psi$ is equivalent to a formula in CNF, and hence $\phi$ is equivalent to a formula of the following form

  $$\bigwedge_i \mathsf{K} \left( \chi_i \vee \mathsf{L}\chi_i' \vee \bigvee_j \mathsf{K}\chi_i^j \right),$$

  since $\mathsf{K}$ distributes over conjunctions. Now, since the formula $\mathsf{K}(\phi \vee \mathsf{K}\psi) \leftrightarrow \mathsf{K}\phi \vee \mathsf{K}\psi$ is a theorem of **S5**, the above formula is equivalent to

  $$\bigwedge_i \left( \mathsf{L}\chi_i' \vee \left( \mathsf{K}\chi_i \vee \bigvee_j \mathsf{K}\chi_i^j \right) \right),$$

  which is in CNF.
- If $\phi = \Diamond\psi$ then, by induction hypothesis, $\phi$ is equivalent to a formula of the form

  $$\Diamond \bigvee_i \left( \chi_i \wedge \mathsf{K}\chi_i' \wedge \bigwedge_j \mathsf{L}\chi_i^j \right).$$

  Since $\Diamond$ distributes over disjunctions in every normal system, the above formula is equivalent to

  $$\bigvee_i \Diamond \left( \chi_i \wedge \mathsf{K}\chi_i' \wedge \bigwedge_j \mathsf{L}\chi_i^j \right).$$

By repeated applications of Lemma 6.1, the above formula is equivalent to

$$\bigvee_i \left( \Diamond\chi_i \wedge \Diamond \mathsf{K}\chi_i' \wedge \bigwedge_j \Diamond \mathsf{L}\chi_i^j \right). \tag{1}$$

Observe that $\Diamond\mathsf{K}\chi'$ is equivalent to $\Diamond\top \wedge \Box\mathsf{K}\chi'$ using Normality, Axioms 7, Lemma 6.2, and 8. Similarly, $\bigwedge_j \Diamond\mathsf{L}\chi_i^j$ is equivalent to $\Diamond\top \wedge \bigwedge_j \mathsf{L}\Diamond\chi_i^j$ using Normality and Lemma 6. Therefore, (1) is equivalent to

$$\bigvee_i \left( \Diamond\chi_i \wedge \Diamond\top \wedge \mathsf{K}\Box\chi_i' \wedge \bigwedge_j \mathsf{L}\Diamond\chi_i^j \right), \tag{2}$$

which is in DNF.

∎

# 3 Canonical Model

The *canonical model* of **SC** is the structure

$$\mathscr{C} = \left( S, \{\xrightarrow{a}, \xrightarrow{\mathsf{L}}\}, v \right),$$

where

$$\begin{array}{l} S = \{s \subseteq \mathscr{L} : s \text{ is } \mathbf{SC}\text{-maximal consistent}\}, \\ s \xrightarrow{a} t \text{ iff } \{\phi \in \mathscr{L} : \Box\phi \in s\} \subseteq t, \\ s \xrightarrow{\mathsf{L}} t \text{ iff } \{\phi \in \mathscr{L} : \mathsf{K}\phi \in s\} \subseteq t, \\ v(A) = \{s \in S : A \in S\}, \end{array}$$

along with the usual satisfaction relation (defined inductively):

$$\begin{array}{lll} s \models_{\mathscr{C}} A & \text{iff} & s \in v(A) \\ s \not\models_{\mathscr{C}} \bot & & \\ s \models_{\mathscr{C}} \neg\phi & \text{iff} & s \not\models_{\mathscr{C}} \phi \\ s \models_{\mathscr{C}} \phi \wedge \psi & \text{iff} & s \models_{\mathscr{C}} \phi \text{ and } s \models_{\mathscr{C}} \psi \\ s \models_{\mathscr{C}} \Box\phi & \text{iff} & \text{for all } t \in S,\ s \xrightarrow{a} t \text{ implies } t \models_{\mathscr{C}} \phi \\ s \models_{\mathscr{C}} \mathsf{K}\phi & \text{iff} & \text{for all } t \in S,\ s \xrightarrow{\mathsf{L}} t \text{ implies } t \models_{\mathscr{C}} \phi. \end{array}$$

We write $\mathscr{C} \models \phi$, if $s \models_{\mathscr{C}} \phi$ for all $s \in S$.

A canonical model exists for all consistent bimodal systems with the normal axiom scheme for each modality. We have the following well known theorems (see [Che80], or [Gol87].)

**Theorem 9 (Truth Theorem)** *For all $s \in S$ and $\phi \in \mathscr{L}$,*

$$s \models_{\mathscr{C}} \phi \qquad \textit{iff} \qquad \phi \in s.$$

**Theorem 10 (Completeness Theorem)** *For all $\phi \in \mathscr{L}$,*

$$\mathscr{C} \models \phi \qquad \textit{iff} \qquad \vdash_{\mathbf{SC}} \phi.$$

We shall now prove some properties of the members of $\mathscr{C}$. We will make use of the following sets: for all $s \in S$

$$s^{\Box} = s \cap \mathscr{L}^{\Box} \qquad s^{\mathsf{K}} = s \cap \mathscr{L}^{\mathsf{K}}$$

and

$$S^{\Box} = \{s^{\Box} : s \in S\} \qquad S^{\mathsf{K}} = \{s^{\mathsf{K}} : s \in S\}.$$

The DNF theorem implies that every maximal consistent theory $s$ of **SC** is determined by the formulas in $\mathscr{L}^{\Box}$ and $\mathscr{L}^{\mathsf{K}}$ it contains, i.e. by $s^{\Box}$ and $s^{\mathsf{K}}$. Moreover, the set $\{\mathsf{K}\phi, \mathsf{L}\phi : \mathsf{K}\phi, \mathsf{L}\phi \in s\}$ is determined by $s^{\mathsf{K}}$ alone (this is the $\mathsf{K}$-case of the DNF theorem.)

The following definition is useful

**Definition 11** Let $P \subseteq \mathscr{L}^{\Box}(\mathscr{L}^{\mathsf{K}})$. We say $P$ is an $\mathscr{L}^{\Box}(\mathscr{L}^{\mathsf{K}})$-*theory* if $P$ is consistent and for all $\phi \in \mathscr{L}^{\Box}(\mathscr{L}^{\mathsf{K}})$ either $\phi \in P$ or $\neg\phi \in P$.

Hence, $s^{\Box}$ is an $\mathscr{L}^{\Box}$-theory and $s^{\mathsf{K}}$ is an $\mathscr{L}^{\mathsf{K}}$-theory. An $\mathscr{L}^{\Box}$-theory and an $\mathscr{L}^{\mathsf{K}}$-theory determine an **SC** maximal consistent theory when their union is consistent because in this case there is a unique maximal extension. To test consistency we have the following lemma.

**Lemma 12** *If $P$ and $S$ are an $\mathscr{L}^{\Box}$- and $\mathscr{L}^{\mathsf{K}}$-theory respectively then $P \cup S$ is consistent if and only if*

$$\textit{if} \quad \phi \in P \quad \textit{then} \quad \mathsf{L}\phi \in S.$$

*Proof.* Suppose that $P \cup S$ is not consistent then there exists $\phi \in P$ and $\{\mathsf{L}\phi_i\}_{i=1}^{n} \subseteq S$ such that

$$\bigwedge_{i=1}^{n} \mathsf{L}\phi_i \to \neg\phi,$$

which implies, since $\mathsf{K}$ distributes over conjunctions,

$$\bigwedge_{i=1}^{n} \mathsf{L}\phi_i \to \mathsf{K}\neg\phi.$$

Therefore $\neg\mathsf{L}\phi \in S$ and $\mathsf{L}\phi \notin S$. The other direction is straightforward because $\phi \to \mathsf{L}\phi$. ∎

It is expected that since $\mathscr{L}^{\Box}$- and $\mathscr{L}^{\mathsf{K}}$-theories determine **SC** maximal consistent sets they will determine their accessibility relations, as well.

**Corollary 13** *Let $T, T'$ be subsets of $\mathscr{L}$. Then,*

1. *$T$ is a maximal consistent subset of $\mathscr{L}^{\Box}$ iff $T \in S^{\Box}$.*
2. *$T$ is a maximal consistent subset of $\mathscr{L}^{\mathsf{K}}$ iff $T \in S^{\mathsf{K}}$.*
3. *If $T \in S^{\Box}$, $T' \in S^{\mathsf{K}}$, then $T \cup T'$ is consistent iff there exists a unique $s \in S$ such that $T = s^{\Box}$ and $T' = s^{\mathsf{K}}$.*

**Lemma 14** *Given $T \in S^{\mathsf{K}}$, if the set*

$$T^R = \{\mathsf{K}\phi : \mathsf{K}\Box\phi \in T\} \cup \{\mathsf{L}\psi : \mathsf{L}\Diamond\psi \in T\}$$

*is consistent then $T^R \in S^{\mathsf{K}}$.*

*Proof.* We need to show maximality so suppose that there exists $\phi \in \mathscr{L}^{\Box}$ such that $\mathsf{K}\phi \notin T^R$ but $T^R \cup \{\mathsf{K}\phi\}$ is consistent. We have that $\mathsf{K}\Box\phi \notin T$ so $\mathsf{L}\Diamond\neg\phi \in T$ and so $\mathsf{L}\neg\phi \in T^R$, a contradiction. This shows that $T^R$ is maximal. ∎

**Lemma 15** *Let $T_1, T_2 \in S^{\Box}$ and $T_1' \in S^{\mathsf{K}}$. If $T_1 \xrightarrow{a} T_2$ and $T_1 \cup T_1'$ is consistent then $T_2 \cup (T_1')^R$ is consistent.*

*Proof.* Suppose $T_2 \cup (T_1')^R$ is inconsistent. Then there exist $\phi \in T_2$, $\mathsf{K}\psi, \mathsf{L}\chi^i \in (T_1')^R$ such that

$$\phi \wedge \mathsf{K}\psi \wedge \bigwedge_i \mathsf{L}\chi_i \vdash \bot$$

which implies

$$\Box(\mathsf{K}\psi \wedge \bigwedge_i \mathsf{L}\chi_i) \vdash \Box\neg\phi$$

so

$$\mathsf{K}\Box\psi \wedge \bigwedge_i \mathsf{L}\Diamond\chi_i \vdash \Box\neg\phi,$$

by Axiom 8. Notice that the right side belongs to $T_1'$ and $\Diamond\phi \in T_1$ which implies that $T_1 \cup T_1'$ is inconsistent, a contradiction. ∎

**Proposition 16** *For all $s,t \in S$,*

| | | |
|---|---|---|
| *a.* $s \overset{a}{\to} t$ | *if and only if* | *i.* $s^{\Box} \overset{a}{\to} t^{\Box}$, |
| | | *ii.* $(s^{\mathsf{K}})^R = t^{\mathsf{K}}$. |
| *b.* $s \overset{\mathsf{L}}{\to} t$ | *if and only if* | $s^{\mathsf{K}} = t^{\mathsf{K}}$. |

*Proof.* For (a), right to left, let $\phi \in t$ then, by the DNF Theorem, $\phi$ has the form

$$\bigvee_i \left( \chi_i \wedge \mathsf{K}\chi_i' \wedge \bigwedge_j \mathsf{L}\chi_i^j \right),$$

where $\chi, \chi', \chi_k^j \in \mathscr{L}^{\Box}$. Then $\Diamond\phi$ has the form

$$\Diamond \bigvee_i \left( \chi_i \wedge \mathsf{K}\chi_i' \wedge \bigwedge_j \mathsf{L}\chi_i^j \right),$$

which is equivalent to

$$\bigvee_i \left( \Diamond\chi_i \wedge \mathsf{K}\Box\chi_i' \wedge \bigwedge_j \mathsf{L}\Diamond\chi_i^j \right),$$

as in the proof of the DNF Theorem. Observe here that, in case $\phi \in \mathscr{L}^{\Box}$, if $\chi_i, \mathsf{K}\chi_i', \mathsf{L}\chi_i^j \in t$ then $\Diamond\chi_i, \mathsf{K}\Box\chi_i', \mathsf{L}\Diamond\chi_i^j \in s$, by *a(i)* and *a(ii)*. Therefore, $s \overset{a}{\to} t$.

For the other direction, *a(i)* is straightforward and for *a(ii)*, use Axioms 7 and 8.

For *(b)*, right to left, we proceed as above. Let $\mathsf{K}\phi \in t$, then, by the DNF Theorem, it has the following form

$$\bigwedge_i \left( \mathsf{L}\chi_i' \vee \bigvee_j \mathsf{K}\chi_i^j \right),$$

where $\chi_i', \chi_i^j \in \mathscr{L}^{\Box}$. Thus $\mathsf{K}\phi \in s$.

The other direction is straightforward by the definition of $\overset{\mathsf{L}}{\to}$. ∎

We will define a subset frame model which is equivalent to the canonical model.

Let $S^{\Box} \dot{\times} S^{\mathsf{K}}$ be the subset of $S^{\Box} \times S^{\mathsf{K}}$ containing all pairs $(T, T')$ such that $T \cup T'$ is consistent. Let $f$ be a map from $S$ to $S^{\Box} \dot{\times} S^{\mathsf{K}}$ defined by $f(s) = (s^{\Box}, s^{\mathsf{K}})$. It is straightforward to show that the map $f$ is 1-1 and onto. Therefore, $f$ has an inverse defined by $f^{-1}(T, T') = s(T, T')$, where $s(T, T') \in S$ is the unique maximal consistent extension of $T \cup T'$ from Corollary 13.3. Therefore, the worlds of the canonical model maybe split in two components that will be used to construct a point and a subset in the following definition.

The *standard subset model* is defined with

$$(X, \mathscr{O}, R, v),$$

where

$$X = \{x_T : T \in S^{\Box}\},$$

$$\mathscr{O} = \{U_T : T \in S^{\mathsf{K}}\},$$

where $U_T = \{x_{T'} : T \cup T' \text{ consistent}\}$,

$$x_T R x_{T'} \quad \text{iff} \quad T \xrightarrow{a} T',$$

and

$$v(A) = \{x_T : A \in T\}.$$

We have the following

**Lemma 17** *For all $U_T \in \mathscr{O}$, we have*

$$U_T^R = U_{T^R}.$$

*Proof.* First we show that $U_T^R \subseteq U_{T^R}$. Suppose $x_{T'} \in U_T^R$ and $x_{T''} R x_{T'}$ with $x_{T''} \in U_T$. We must show that $x_{T'} \in U_{T^R}$. It is enough to show that $T'' \cup T^R$ is consistent. This follows from Lemma 15.

For the other direction suppose $x_{T'} \in U_{T^R}$. We must find $x_{T''} \in U_T$ with $x_{T''} R x_{T'}$. To this end, let

$$A = \{\Diamond \phi : \phi \in T'\} \cup T.$$

The set $A$ is consistent. For if not, there exist $\phi \in T'$ and $\mathsf{K}\psi, \mathsf{L}\chi^i \in T$ where $\psi, \chi_i \in \mathscr{L}^{\Box}$, such that

$$\Diamond \phi \wedge \mathsf{K}\psi \wedge \bigwedge_i \mathsf{L}\chi^i \vdash \bot.$$

So

$$\mathsf{K}\psi \wedge \bigwedge_i \mathsf{L}\chi^i \vdash \Box \neg \phi.$$

The above implies

$$\mathsf{K}\psi \wedge \bigwedge_i \mathsf{L}\chi^i \vdash \mathsf{K}\Box \neg \phi.$$

The right side belongs to $T$ and so does the left side, because $T$ is a maximal consistent subset of $\mathscr{L}^{\mathsf{K}}$ and $\mathsf{K}\Box\neg\phi \in \mathscr{L}^{\mathsf{K}}$. By the definition of $T^R$, we have that $\mathsf{K}\neg\phi \in T^R$ and therefore $T' \cup T^R$ is inconsistent, a contradiction. Extend $A$ to a maximal consistent subset $s$ of $\mathscr{L}$. We have $x_{s^{\Box}} R x_{T'}$ and $s^{\mathsf{K}} = T$ so $s^{\Box}$ is consistent with $T$ and therefore $x_{s^{\Box}} \in U_T$. ∎

As a corollary, $\mathscr{O}$ is $R$-closed and so the canonical subset model is well defined. It remains to prove completeness:

**Theorem 18** *For all $(x_T, U_{T'}) \in X \dot{\times} \mathscr{O}$, we have*

$$x_T, U_{T'} \models \phi \quad \textit{iff} \quad \phi \in s(T, T').$$

*Proof.* By structural induction on $\phi$.

- If $\phi$ is $A$, where $A$ is atomic then

$$x_T, U_{T'} \models A \quad \text{iff} \quad A \in T \quad \text{iff} \quad A \in T \cup T' \quad \text{iff} \quad A \in s(T, T').$$

- The boolean cases are straightforward.

- Suppose $\phi$ is of the form $\Box\psi$.

  Assume that $x_T, U_{T'} \models A$, then for all $T''$ such that $T \xrightarrow{a} T''$ we have $x_{T''}, U_{T'^R} \models \psi$. Now, suppose $s(T,T') \xrightarrow{a} s$. We need to show that $\psi \in s$. We have by Proposition 16 that $T \xrightarrow{a} s^{\Box}$ so $x_T R x_{s^{\Box}}$, and $T'^R = s^{\mathsf{K}}$ so $U_{T'^R} = U_{s^{\mathsf{K}}}$. Therefore, $x_{s^{\Box}}, U_{s^{\mathsf{K}}} \models \psi$ by our assumption, and so $\psi \in s$ by the induction hypothesis.

  For the other direction, suppose $\Box\psi \in s(T,T')$. We must show that $x_T, U_{T'} \models \Box\phi$. To this end, let $x_T R x_{T''}$. We have $T \xrightarrow{a} T''$. By Lemma 15 we have that $T'' \cup (T')^R$ is consistent and so $s(T,T') \xrightarrow{a} s(T'',(T')^R)$, by Proposition 16. Therefore $x_{T''}, U_{(T')^R} \models \psi$, by the induction hypothesis.

- Suppose $\phi$ is of the form $\mathsf{K}\psi$.

  Assume that $\mathsf{K}\psi \in s(T,T')$. We need to show that $x_T, U_{T'} \models \mathsf{K}\psi$. Let $x_{T_1} \in U_{T'}$. By Proposition 16, we have $s(T_1,T') \xrightarrow{\mathsf{L}} s(T,T')$. Therefore, $\psi \in s(T_1,T')$ so $x_{T_1}, U_{T'} \models \psi$ by the induction hypothesis.

  Now assume that $x_T, U_{T'} \models \mathsf{K}\psi$. We must show $\mathsf{K}\psi \in s(T,T')$. Let $s(T,T') \xrightarrow{\mathsf{L}} s$. By Proposition 16, $T' = s^{\mathsf{K}}$ so $x_{s^{\Box}}, U_{T'} \models \psi$ by our assumption. The latter implies $\psi \in s(s^{\Box},T') = s$ by the induction hypothesis.

∎

# 4 Decidability

At this section, we will show that the logic **SC** is decidable by showing that it possesses the finite model property: if a formula is satisfiable then it is satisfiable in a finite subset model, i.e., a subset model that has a finite number of points and therefore a finite number of subsets.

The proof relies upon both the normal form theorem of the previous section as well as a filtration.

To this end, let $\phi$ be a formula and $(X,\mathcal{O},R,i)$ a subset model in which $\phi$ is satisfied, that is, there exist $x \in X$ and $U \in \mathcal{O}$ such that $x,U \models \phi$. The Normal form theorem allow us to assume that $\phi$ is in DNF, i.e. a has the form

$$\bigvee_i \left( \chi_i \wedge \mathsf{K}\chi_i' \wedge \bigwedge_j \mathsf{L}\chi_i^j \right),$$

where $\chi, \chi', \chi_k^j \in \mathcal{L}^{\Box}$. Denote the set $\{\chi_i, \chi_i', \chi_i^j : i = 1,\ldots,n\}$ with $\mathcal{L}_\phi^{\Box}$ and $\{\mathsf{L}\chi_i^j : i = 1,\ldots,n\}$ with $\mathcal{L}_\phi^{\mathsf{K}}$.

We will now define a filtration on $(X,\mathcal{O},i)$ as follows. Let $\sim_\phi$ be an equivalence relation on $X$ defined by $x \sim_\phi y$, for $x,y \in X$, when $x,U \models \chi$ iff $y,U \models \chi$ for all $\chi \in \mathcal{L}_\phi^{\Box}$ and some $U \in \mathcal{O}$. Note here that a straightforward induction shows that the satisfaction of a formula $\chi$ in $\mathcal{L}^{\Box}$ is independent of the subset, that is, $x,V \models \chi$ for some $V \in X$ iff $y,U \models \chi$ for all $U \in X$. Denote the equivalence class of $x$ under $\sim_\phi$ with $x_\phi$, the set $\{x_\phi : x \in U\}$ with $U_\phi$, and the set of all equivalence classes with $X_\phi$. Now consider the subset model $(X_\phi, \mathcal{P}(X_\phi), R_\phi, i_\phi)$, where, for all atomic $A$ in $\mathcal{L}_\phi^{\Box}$,

$$i_\phi(A) = \{x_\phi : x \in i(A)\}$$

and, for all $\Box b \in \mathcal{L}_\phi^{\Box}$ and some $U,V \in \mathcal{O}$,

$$x_\phi R_\phi y_\phi \text{ iff } x,U \models \Box b \text{ then } y,V \models b.$$

Observe that the powerset $\mathcal{P}(X_\phi)$ is $R_\phi$-closed.

We have the following

**Lemma 19** *For all $\psi \in \mathcal{L}_\phi^{\Box}$, we have*

$$x,U \models \psi \text{ iff } x_\phi, U_\phi \models \psi.$$

*Proof.* The atomic case follows from the definition of $i_\phi$. The boolean cases are straightforward. Now, suppose $\psi$ is of the form $\Box\chi$. Let $x,U \models \Box\chi$. To show that $x_\phi, U_\phi \models \Box\chi$. Let $x_\phi R_\phi y_\phi$. By definition of $R_\phi$, we have $y,V \models \chi$ for some $V \in \mathcal{O}$. By induction hypothesis, we have $y_\phi, V_\phi \models \chi$. As mentioned above $y_\phi, W \models \chi$ for all $W \subseteq X_\phi$ since $\chi \in \mathcal{L}_\phi^\Box$. In particular, $y_\phi, U_\phi^{R_\phi} \models \chi$. For the other direction, suppose $x_\phi, U_\phi \models \Box\chi$ and let $xRy$. We have $x_\phi R_\phi y_\phi$, and so $y_\phi, U_\phi^{R_\phi} \models \chi$. As $\chi \in \mathcal{L}_\phi^\Box$, we have $y_\phi, (U^R)_\phi \models \chi$ ($y_\phi \in (U^R)_\phi$ because $y \in U^R$). By induction hypothesis, $y, U^R \models \chi$. ∎

This extends to all subformulas of $\phi$ by the following

**Lemma 20** *For all $\psi$, where $\psi$ is a subformula of $\phi$, we have*

$$x,U \models \psi \textit{ iff } x_\phi, U_\phi \models \psi.$$

*Proof.* If $\psi \in \mathcal{L}_\phi^\Box$, the lemma follows from the previous lemma. The boolean cases are straightforward. Now, suppose $\psi$ belongs to $\mathcal{L}_\phi^{\mathsf{K}}$, i.e. is of the form $\mathsf{L}\chi$ where $\chi \in \mathcal{L}_\phi^\Box$. Let $x,U \models \mathsf{L}\chi$. To show that $x_\phi, U_\phi \models \mathsf{L}\chi$. There exists $y \in U$ such that $y,U \models \chi$. By induction hypothesis we have $y_\phi, U_\phi \models \chi$, so $x_\phi, U_\phi \models \mathsf{L}\chi$. For the other direction, suppose $x_\phi, U_\phi \models \mathsf{L}\chi$. There exists $y_\phi \in U_\phi$ such that $y_\phi, U_\phi \models \chi$. By the definition of $U_\phi$ there exists $z \in U$ such that $z_\phi = y_\phi$. We have $z_\phi, U_\phi \models \chi$, so, by induction hypothesis, $z, U \models \chi$ and therefore $x, U \models \mathsf{L}\chi$. ∎

As a consequence, the logic **SC** satisfies the finite model property with respect to the class of **SC**-models. The main result of this section follows.

**Corollary 21** *The logic* **SC** *is decidable.*

# 5 Conclusion

We have presented a variant of the Moss-Parikh Subset Logic that handles arbitrary changes along with a completeness and decidability result. Our presentation has made use of a single modality but extending the language, semantics, and subsequent results to a multi-modal setting is straightforward. For the sake of completeness we will briefly mention how this can be done but we will omit all details.

First, the language will be augmented with a set $\mathsf{Act}$ of symbols corresponding to sorts of changes. A change can be a result of an action but not necessarily so (for example, time passing). As a result, we need to include in the language formulas of the form $[a]\phi$. On the semantics side, models will be equipped with the set $\{R_a : a \in \mathsf{Act}\}$ of binary relations on $X$. For each $U \in \mathcal{O}$ and $a \in \mathsf{Act}$, let $U^{R_a}$ be the set of the elements accessible from $U$, that is, the set $\{y : (x,y) \in R_a, x \in U\}$. The set $\mathcal{O}$ will be called *$R_a$-closed* if whenever $U \in \mathcal{O}$ then $U^{R_a} \in \mathcal{O}$. If $\mathcal{O}$ is $R_a$-closed for each $a \in \mathsf{Act}$, then the triple $\langle X, \{R_a\}_{a \in \mathsf{Act}}, \mathcal{O} \rangle$ will be called a *action subset frame* and proceed similarly for the definition of the model. Satisfaction now will include the case

$$x,U \models_{\mathcal{M}} [a]\phi \quad \text{if} \quad \text{for all } y \in X \text{ such that } (x,y) \in R_a,\ y, U^{R_a} \models_{\mathcal{M}} \phi.$$

Now, all results including decidability and the normal form theorem lift to the extended language in a straightforward way as actions do not interact with each other. This extended language allow us to express the original Moss-Parikh restriction modality using a modality $[U]$ for each $U \in \mathcal{O}$, whose semantics are given by the relation

$$R_U = \{(x,x) : x \in U\}.$$

The update operator is an interesting addition to the already extensive arsenal of subset logic. We believe that such an addition is very useful as a building block to an epistemic logic that handles change in various forms. As an example, update can be combined with the public announcement operator of dynamic epistemic logic.

The combination of propositional dynamic logic with a logic of knowledge (**PDL** + **K5**) has been extensively studied in [ST08]. These results do not carry over in our system because of the interaction axiom

$$\langle a \rangle \mathsf{K}\phi \to \mathsf{K}[a]\phi$$

which is stronger than the ones (*NL* and *CR*) considered in [ST08]. Adding a calculus of action in the manner of dynamic logic, or interpreting the modalities in a temporal context is perhaps the most promising extension of this logic.

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