Dominance plausible rule and transitivity

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Abstract. In qualitative decision theory, a very natural way for defining preference relations over the policies (acts) -functions from a set S of states to a set X of consequences- is by using the so called *Dominance Plausible Rule*. In this context we need a relation $>$ over X and a relation $\sqsupseteq$ over $\mathcal{P}(S)$. Then we define $\succeq$ as follows: $f \succeq g \;\Leftrightarrow\; [f > g] \sqsupseteq [g > f]$, where $[f > g]$ denotes the set $\{s \in S : f(s) > g(s)\}$. In many cases $>$ is a modular relation and $\sqsupseteq$ is a total preorder. A quite rational and desirable property for the relation over the policies is the transitivity. In general, the relation $\succeq$ defined by the Dominance Plausible Rule is not transitive in spite of the transitivity of $\sqsupseteq$. In this work we characterize the properties of the relation $\sqsupseteq$ forcing the relation $\succeq$ over the policies to be transitive. All this under the hypothesis of modularity of the relation $>$.

1 Introduction

Given a set of states S and a set of consequences X, a policy (act) is a function $f : S \longrightarrow X$. We denote by X^S the set of policies. One of the most important problems in Decision Theory is to know which is the best policy. In order to decide which is the best policy when we have a probability over the events (sets of states) and a utility function $u : X \longrightarrow \mathbb{R}$, one can classify the policies via the expected value of the functions $u \circ f$ where f is a policy. The best policies are those maximizing the expected value. Thus, given a utility function u over X and a probability function p over S, we can define the *expected utility* of f as

$$UE(f) = \sum_{s \in S} p(s) u(f(s))$$

(that is, the expected value of $u \circ f$). Then, very naturally, we classify the policies in the following way

$$f \succeq g \Leftrightarrow UE(f) \geq UE(g) \quad (1)$$

Savage [5] proved that if the relation between the policies $\succeq$ satisfies some axioms then there is a probability function p over S and also a utility function u over X such that the equation 1 holds, *i.e.* the decision relation $\succeq$ over the policies can be defined via the expected utility when that relation obeys to certain rationality criteria. Savage's framework makes sense for an infinite set of states. Now, when we consider a finite set of states, Savage's axioms do not hold. Moreover, the expected utility is sensitive to small variations. For instance, suppose we have three states s_1, s_2 and s_3 where the states s_1 and s_2 are equally plausible and, in turn, more plausible than s_3. We must decide which policy between f_1 and f_2 is the best, and our decision will be founded using the utility expected model. Suppose that $S = \{s_1, s_2, s_3\}$, and f_1 and f_2 are defined as follows:

	s_1	s_2	s_3
f_1	x_1	x_3	x_2
f_2	x_5	x_6	x_4

Consider the probabilities p_1 and p_2 over S given by

	s_1	s_2	s_3
p_1	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{1}{5}$
p_2	$\frac{9}{20}$	$\frac{9}{20}$	$\frac{1}{10}$

Let u be the utility function for the consequences given by the following table:

	x_1	x_2	x_3	x_4	x_5	x_6
u	10	60	36	50	20	30

If we classify the policies f_1 and f_2 using the expected utility with the probability p_1 and the utility function u, we have $UE(f_1) = 30.4$ and $UE(f_2) = 30$; thus $UE(f_1) > UE(f_2)$ and therefore $f_1 \succ f_2$.

But if we calculate the expected utility using the probability p_2 and the utility function u, we have $UE(f_1) = 26.7$ and $UE(f_2) = 27.5$; thus $UE(f_1) < UE(f_2)$ and therefore $f_1 \prec f_2$.

The previous example shows that the quantitative framework is very sensitive to small variations in the inputs. This is one of reasons to look for pure qualitative frameworks more robust and more appropriate for the finite case. In this direction some recent works have been developed, for instance Dubois et al. [2, 3].

One of the main contributions of Dubois et al. work (we will call this approach the DFPP framework) is the axiomatic characterization *à la* Savage for the relations defined by the Dominance Plausible Rule (RDP) steaming from a relation over the consequences and a possibilistic relation over the events.

More precisely, we have a relation $>$ over X and a relation $\sqsupseteq$ over $\mathcal{P}(S)$. Then we define $\succeq$ as follows:

$$f \succeq g \quad \Leftrightarrow \quad [f > g] \sqsupseteq [g > f] \tag{2}$$

where $[f > g]$ denotes the set $\{s \in S : f(s) > g(s)\}$. Usually $>$ is a modular[3] relation and $\sqsupseteq$ is a total pre-order[4].

The definition given by 2 captures, in the finite case, the definition given by 1. The intended meaning of definition 2 is the following one: an agent should always choose action f over action g if she considers the event that f leads to a strictly preferable outcome than g more likely than the event that g leads to a strictly preferable outcome than f.

The definition given by 2 is quite natural and, in some cases (for instance when the relation $\sqsupseteq$ is the intersection of a family of possibilistic relations) we have an axiomatic characterization which is very similar to that of Savage's (see [2, 3]).

We would like the relation $\succeq$ defined over the policies by 2 to have good properties. In particular, it should have a minimally rational behavior. It is natural to expect that certain properties of the relation $\sqsupseteq$ will induce a good behavior of the relation $\succeq$ defined over the policies

One of the properties of rationality that one would like to have is the transitivity of the relation $\succeq$ defined over the policies. In this work we investigate how this property is reflected in the relation $\sqsupseteq$. Actually, we give the properties of the relation $\sqsupseteq$ characterizing the transitivity of the relation $\succeq$ defined by 2.

2 Preliminares

Recall that a total preorder R over a set C is a binary relation over C that is transitive and complete (in particular, it is reflexive). In this work we assume that $\geq_X$ is a total pre-order over X and $\sqsupseteq$ is any binary relation over $\mathcal{P}(S)$.

Any relation $\sqsupseteq$ determines two relations $\sqsupset$ and $\sim$, the associated strict relation and the associated indifference relation:

$$A \sqsupset B \Leftrightarrow A \sqsupseteq B \quad \& \quad B \not\sqsupseteq A$$
$$A \sim B \Leftrightarrow A \sqsupseteq B \quad \& \quad B \sqsupseteq A$$

Notice that the strict relation is antisymmetrical. The relation $\sqsupseteq$ is said to be consistent[5] if $\forall A \in \mathcal{P}(S)$ with $A \neq \emptyset$, $A \sqsupseteq \emptyset$.

$(X^S, \succeq)$ denotes the set of policies equipped with a preference relation $\succeq$, with strict preference $\succ$ and indifference $\backsim$. Given $f, g \in X^S$ and $A \in \mathcal{P}(S)$, we denote by fAg the function taking the value $f(s)$ if $s \in A$ and $g(s)$ if $s \in A^c$. We define $[f >_X g] = \{s \in S : f(s) >_X g(s)\}$. Notice that the sets $[f >_X g]$,

[3] Sometimes in the literature these relations are called *weak orders* (see for instance [4]). A relation R over X is modular iff R is transitive and if xRy and $\neg(yRz)$ and $\neg(zRy)$ then xRz.

[4] A total pre-order is a transitive and total relation:

[5] Our consistency notion is weaker than the notion found in [2] where they ask $S \sqsupseteq A$ for any A.

$[g >_X f]$ and $[f \sim_X g]$ are mutually disjoint and $[f \geq_X g] = [g >_X f]^c$.
Given $\geq_s$, a total pre-order over S, Dubois et.al [2] define a possibilistic relation[6] $\sqsupseteq_\Pi$ over $\mathcal{P}(S)$ steaming from $\geq_s$ as follows:

$$A \sqsupseteq_\Pi B \Leftrightarrow \ \forall x \in B \ \exists y \in A \ \ y \geq_s x$$

Definition 1 *Let $\geq_X$ be a total pre-order over X and $\sqsupseteq$ a relation over $\mathcal{P}(S)$. We say that the relation $\succeq$ over X^S is defined by the Dominance Plausible Rule (*DPR*) with $(\geq_X, \sqsupseteq)$ when the following equivalence holds:*

$$f \succeq g \Leftrightarrow [f >_X g] \sqsupseteq [g >_X f]$$

It is easy to see that if $\sqsupseteq$ is a total pre-order over $\mathcal{P}(S)$, then

$$f \succ g \Leftrightarrow [f >_X g] \sqsupset [g >_X f]$$

and

$$f \backsim g \Leftrightarrow [f >_X g] \sim [g >_X f]$$

Properties like transitivity and totality are desirable, in some contexts, for the relation $\succeq$. Dubois et al. [3], prove that if $\succeq$ is defined by DPR with $(\geq_X, \sqsupseteq)$, where $\geq_X$ is a total pre-order over X and $\sqsupseteq$ is a relation over $\mathcal{P}(S)$ which is reflexive, non trivial $(S \sqsupset \emptyset)$ and consistent, then $\succeq$ is reflexive and total over the constant policies. Moreover, it is easy to see that if $\sqsupseteq$ is total over $\mathcal{P}(S)$, then $\succeq$ is total. Given $\geq_s$ a total pre-order over S, we can consider a lifting $\sqsupseteq$ generated by $\geq_s$ (*i.e.* relations $\sqsupseteq$ over $\mathcal{P}(S)$ satisfying the extension rule (E): $x \geq_s y \Leftrightarrow \{x\} \sqsupseteq \{y\}$).

Two very natural liftings are $\sqsupseteq_\Pi$, already defined, and the leximin, $\sqsupseteq^L_{min}$, (see [1]) for definition of which we need some more notation. Suppose $|S| = n$ and consider $V\downarrow$ the set of all vectors of size less or equal to n, the inputs of which are elements of S; there are not repetitions of the inputs and finally they are ordered in decreasing manner by $\geq_s$. That is, given $k \leq n$, $\boldsymbol{a} = (a_1, \cdots, a_k) \in V\downarrow$ iff, for all $1 \leq i, j \leq k$ with $i \neq j$, $a_i \neq a_j$ and $\forall\, 1 \leq i \leq k-1 \ a_i \geq_s a_{i+1}$. Now, given $\boldsymbol{a}$, $\boldsymbol{a'} \in V\downarrow$ of length m with $m \leq n$, we define the following relation:

$$\boldsymbol{a} \equiv \boldsymbol{a'} \Leftrightarrow a_i \sim_s a'_i, \quad \forall i = 1, \cdots, m.$$

Next we define $\succeq^L_{min}$ over $V\downarrow$:

$$\boldsymbol{a} \succeq^L_{min} \boldsymbol{b} \Leftrightarrow \begin{cases} \boldsymbol{a} \equiv \boldsymbol{b} \text{ or} \\ \exists k \in \{1, \cdots, min\{|\boldsymbol{a}|, |\boldsymbol{b}|\}\}, \text{ such that } \forall i < k \ a_i \sim_s b_i \text{ y } a_k >_s b_k \text{ or} \\ |\boldsymbol{a}| > |\boldsymbol{b}| \text{ and } \forall i \in \{1, \cdots, |\boldsymbol{b}|\}, \ a_i \sim_s b_i. \end{cases}$$

Let $A \in \mathcal{P}(S)$ and suppose that $|A| = k$. The set of vectors in $V\downarrow$ the length of which is k with inputs in A will be denoted by $R(A)$, that is

$$R(A) = \{\boldsymbol{a} \in V\downarrow\!: \ |\boldsymbol{a}| = k \text{ and the inputs of } \boldsymbol{a} \text{ are in } A\}.$$

[6] Actually our definition corresponds to a *non dogmatic relation* in the sense of [2], *i.e.* the empty set is the minimal element.

Now we define $\sqsupseteq^L_{min}$ over $\mathcal{P}(S)$ as follows:

$$A \sqsupseteq^L_{min} B \Leftrightarrow \forall \boldsymbol{b} \in R(B)\ \exists \boldsymbol{a} \in R(A)\ \boldsymbol{a} \succeq^L_{min} \boldsymbol{b}$$

If we define $\succeq$ by DPR with $(\geq_X, \sqsupseteq^L_{min})$ or $(\geq_X, \sqsupseteq_\Pi)$, we have that $\succeq$ is reflexive and total. However, we will see that $\succeq$ is not transitive, and this in spite of the transitivity of the relations $\sqsupseteq^L_{min}$ and $\sqsupseteq_\Pi$.

Dubois et al. [2] also prove that if $\succeq$ is defined by DPR with $(\geq_X, \sqsupseteq_\Pi)$, then $\succ$ is transitive, even if the relation $\succeq$ is not transitive.

The next example shows that if $\succeq$ is defined by DPR with $(\geq_X, \sqsupseteq^L_{min})$, then $\succeq$ is not transitive. Actually, the relation $\succ$ is not transitive.

Example 2 *Consider* $S = \{a_1, a_2, a_3, b_1, b_2, c_1\}$. *Suppose* $a_1 \sim_s a_2 \sim_s a_3 >_s b_1 \sim_s b_2 >_s c_1$. *Suppose there are* $x, y, z \in X$ *such that* $x >_X y >_X z$. *Consider* $f, g, h \in X^S$ *defined by*

	a_1	a_2	a_3	b_1	b_2	c_1
f	x	z	y	y	y	y
g	z	y	y	x	y	z
h	y	x	z	x	z	y

We have $[f >_X g] = \{a_1, c_1\}$, $[g >_X f] = \{a_2, b_1\}$, $[f >_X h] = \{a_1, a_3, b_2\}$, $[h >_X f] = \{a_2, b_1\}$, $[g >_X h] = \{a_3, b_2\}$ *and* $[h >_X g] = \{a_1, a_2, c_1\}$.
If $\succeq$ *is defined by DPR with* $(\geq_X, \sqsupseteq^L_{min})$, *then*

$$[g >_X f] \sqsupset^L_{min} [f >_X g] \Leftrightarrow g \succ f,$$

$$[f >_X h] \sqsupset^L_{min} [h >_X f] \Leftrightarrow f \succ h,$$

$$[h >_X g] \sqsupset^L_{min} [g >_X h] \Leftrightarrow h \succ g.$$

Thus, if $\succ$ *were transitive we would have* $f \succ g$. *But this together with* $g \succ f$ *contradicts the antisymmetry of* $\succ$. *Therefore the relation* $\succ$ *is not transitive.*

We postpone the proof that $\succeq$ is not transitive when it is defined by DPR with a possibilistic relation. This result is a consequence of Proposition 5. Similarly we postpone the proof that the strict relation $\succ$ associated to $\succeq$ is indeed transitive when $\succeq$ is defined by DPR with a possibilistic relation. The last claim will be a consequence of the main result of this work (Theorem 11).

3 Characterization

We start this section by stating a property on the relations $\sqsupseteq$ defined over $\mathcal{P}(S)$. We will show that this property is a necessary condition in order to have transitivity for the relation $\succeq$ defined over the policies by DPR.

Property T: Let A, B, C, D be in $\mathcal{P}(S)$ such that $A \cap B = \emptyset$ and $C \cap D = \emptyset$. If $A \sqsupseteq B$ and $C \sqsupseteq D$, then $(A \cup C) \setminus (B \cup D) \sqsupseteq (B \cup D)$.

Consider a strict possibilistic relation $\sqsupset_{\Pi}$ and the relation $\succ$ over the policies defined by DPR with $(\geq_X, \sqsupset_{\Pi})$. We know, by the works of Dubois and coleagues [2, 3], that the relation $\succ$ is transitive. So, by a result we will prove below (Proposition 5), the relation $\sqsupset_{\Pi}$ satisfies Property T. This will be our first example of a relation satisfying T.

Next we will give another example of a relation $\sqsupseteq$ over $\mathcal{P}(S)$ satisfying the Property T. We will also see that the relation $\succeq$ defined by DPR with $(\geq_X, \sqsupseteq)$ will be transitive. Unfortunately, as we will see below, this is not always the case, that is, Property T is a necessary condition for transitivity but it is not a sufficient condition.

Example 3 *Consider $S = \{s_1, s_2, s_3\}$ and the total pre-order $\sqsupseteq$ over $\mathcal{P}(S)$ defined in the following way:*

$$\{s_1, s_2, s_3\} \sqsupset \{s_1, s_3\} \sim \{s_2, s_3\} \sim \{s_3\} \sqsupset \{s_1, s_2\} \sim \{s_2\} \sqsupset \{s_1\} \sqsupset \emptyset$$

By the hypotheses of Property T the pairs to compare are the following ones: $\{s_1, s_2, s_3\} \sqsupset \emptyset$, $\{s_1, s_3\} \sqsupset \{s_2\}$, $\{s_1, s_3\} \sqsupset \emptyset$, $\{s_2, s_3\} \sqsupset \{s_1\}$, $\{s_2, s_3\} \sqsupset \emptyset$, $\{s_3\} \sqsupset \{s_1, s_2\}$, $\{s_3\} \sqsupset \{s_2\}$, $\{s_3\} \sqsupset \{s_1\}$, $\{s_3\} \sqsupset \emptyset, \{s_1, s_2\} \sqsupset \emptyset$, $\{s_2\} \sqsupset \{s_1\}$, $\{s_2\} \sqsupset \emptyset$, $\{s_1\} \sqsupset \emptyset$.
An exhaustive analysis allows to show that the relation $\sqsupseteq$ satisfies Property T. For instance if we take $\{s_2, s_3\} \sqsupset \{s_1\}$ with all of the possible pairs, after the hypotheses of Property T, we obtain the following $\{s_3\} \sqsupset \{s_1\}$, $\{s_3\} \sqsupset \{s_1, s_2\}$ y $\{s_2, s_3\} \sqsupset \{s_1\}$ all of which are true.
Now suppose that $X = \{x, y\}$ and $x >_X y$. Consider all the functions of X^S. They are defined in the following table

	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8
s_3	x	x	x	x	y	y	y	y
s_2	x	x	y	y	x	x	y	y
s_1	x	y	x	y	x	y	x	y

When $\succeq$ is defined by DPR with $(\geq_X, \sqsupseteq)$. We obtain

$$f_1 \succ f_2 \succ f_3 \succ f_4 \succ f_5 \succ f_6 \succ f_7 \succ f_8,$$

Thus, $\succeq$ is transitive. Moreover, $\succeq$ is linear.

The following example shows that the relation $\sqsupseteq_{min}^{L}$ does not satisfy Property T.

Example 4 *Put $S = \{a_1, a_2, b_1\}$; suppose $a_1 \sim_s a_2 >_s b_1$. Consider $A = \{a_1\} = C$, $B = \{a_2\}$ and $D = \{b_1\}$. Note that $A \cap B = \emptyset$, $C \cap D = \emptyset$ and $A \sim_{max}^{U} B$, $C \sqsupset_{min}^{L} D$. Thus $A \sqsupseteq_{min}^{L} B$ and $C \sqsupseteq_{min}^{L} D$. Moreover, $B \cup D = \{a_2, b_1\}$ and $(A \cup C) \setminus (B \cup D) = \{a_1\}$. However, $B \cup D \sqsupset_{min}^{L} (A \cup C) \setminus (B \cup D)$. Therefore $\sqsupseteq_{min}^{L}$ does not satisfy Property T.*

The next result states that under certain assumptions, if a relation over the policies defined by the rule of plausible dominance is transitive, then the relation over the events satisfies Property T. More precisely:

Proposition 5 *Let $\geq_X$ be a total pre-order over X and let $\sqsupseteq$ be any relation over $\mathcal{P}(S)$. Suppose there are $x, y, z \in X$ with $x >_X y >_X z$. Let $\succeq$ be the relation over X^S defined by DPR with $(\geq_X, \sqsupseteq)$. If $\succeq$ is transitive, then $\sqsupseteq$ satisfies Property T.*

Proof. Let A, B, C, D in $\mathcal{P}(S)$ be such that $A \cap B = \emptyset$, $C \cap D = \emptyset$, $A \sqsupseteq B$ and $C \sqsupseteq D$. We want to show that $(A \cup C) \setminus (B \cup D) \sqsupseteq B \cup D$. Let x, y, z be in X such that $x >_X y >_X z$. Define $f, g, h \in X^S$ as follows:

	E	F	G	H	I	J	K	L	M
f	x	y	x	z	z	z	y	y	x
g	y	z	y	x	y	y	y	y	x
h	z	x	y	y	x	y	z	x	x

where $E = A \cap C$, $F = A \cap D$, $G = A \cap C^c \cap D^c$, $H = B \cap C$, $I = B \cap D$, $J = B \cap C^c \cap D^c$, $K = C \cap A^c \cap B^c$, $L = D \cap A^c \cap B^c$, $M = A^c \cap B^c \cap C^c \cap D^c$.
Note that
$[f >_X g] = E \cup F \cup G = A$, $\quad [g >_X f] = H \cup I \cup J = B$,
$[g >_X h] = E \cup H \cup K = C$, $\quad [h >_X g] = F \cup I \cup L = D$,
$[f >_X h] = E \cup G \cup K = (A \cup C) \setminus (B \cup D)$ and $[h >_X f] = F \cup H \cup I \cup J \cup L = B \cup D$.
By DPR,

$$f \succeq g \quad \text{and} \quad g \succeq h,$$

Thus, by transitivity of $\succeq$,

$$f \succeq h,$$

again by DPR

$$(A \cup C) \setminus (B \cup D) \sqsupseteq B \cup D.$$

That is, the relation $\sqsupseteq$ satisfies Property T.

Corollary 6 *The relation $\succeq$ defined by DPR with $(\geq_X, \sqsupseteq_\Pi)$ is not transitive.*

Proof. By Proposition 5, it is enough to show that the relation $\sqsupseteq_\Pi$ does not satisfy Property T. Put $S = \{a, b, c\}$ and suppose $a \sim_s b >_s c$. Consider $A = \{a, c\}$, $B = \{b\} = C$ and $D = \{a\}$. Note that $A \cap B = \emptyset$, $C \cap D = \emptyset$ and $A \sim_\Pi B$, $C \sim_\Pi D$. Thus $A \sqsupseteq_\Pi B$ and $C \sqsupseteq_\Pi D$. Moreover, $B \cup D = \{a, b\}$ and $(A \cup C) \setminus (B \cup D) = \{c\}$ but $B \cup D \sqsupset_\Pi (A \cup C) \setminus (B \cup D)$. Therefore, $\sqsupseteq_\Pi$ does not satisfy Property T.

Let us note that for Example 4 and Proposition 5 we can conclude that the relation $\succeq$ defined by DPR with $(\geq_X, \sqsupseteq^L_{min})$ is not transitive.

Now we will see that the fact that $\sqsupseteq$ satisfies Property T is not enough for the transitivity of a relation defined by DPR with $(\geq_X, \sqsupseteq)$. In order to do that, we need the following Lemma:

Lemma 7 *Let $\geq_X$ be a total pre-order over X such that there are $x, y, z \in X$ satisfying*
$x >_X y >_X z$. Let $\sqsupseteq$ be a total pre-order over $\mathcal{P}(S)$ such that $\forall A \in \mathcal{P}(S)$, $A \sqsupseteq \emptyset$. If there are $A, B, C, D \in \mathcal{P}(S)$ such that $A \cap B = \emptyset$, $C \cap D = \emptyset$, $(A \cup C) \setminus (B \cup D) = \emptyset$ and $A \sqsupset B$ and $C \sqsupseteq D$, then $\succeq$ is not transitive.

Proof. Assume $\succeq$ is transitive. Let A, B, C, D be in $\mathcal{P}(S)$ such that $A \cap B = \emptyset$, $C \cap D = \emptyset$, $(A \cup C) \setminus (B \cup D) = \emptyset$. These three hypotheses entail easily $A \subseteq D$ and $C \subseteq B$. Suppose $A \sqsupset B$ and $C \sqsupseteq D$. Let f, g, h be in X^S, defined by the following table:

	A	C	$B \cap D$	$B \cap C^c \cap D^c$	$D \cap A^c \cap B^c$	$B^c \cap D^c$
f	y	z	z	z	y	x
g	z	x	y	y	y	x
h	x	y	x	y	x	x

Note that
$[f >_X g] = A$, $[g >_X f] = B$, $[g >_X h] = C$, $[h >_X g] = D$, $[f >_X h] = \emptyset$ and $[h >_X f] = B \cup D$. Since $A \sqsupset B$ and $C \sqsupseteq D$, by DPR,

$$f \succ g \quad \text{and} \quad g \succeq h.$$

Because $B \cup D \sqsupseteq \emptyset$, necessarily $[h >_X f] \sqsupseteq [f >_X h]$. Thus, $h \succeq f$. By transitivity, $g \succeq f$ and by DPR, $B \sqsupseteq A$, which contradicts the hypothesis $A \sqsupset B$.

Next example shows that the converse of Proposition 5 does not hold.

Example 8 *Consider $S = \{s_1, s_2, s_3\}$ and $\sqsupseteq$ the relation over $\mathcal{P}(S)$ defined as follows:*

$$\{s_1\} \sqsupset \{s_3\} \sim \{s_2\} \sim \{s_2, s_3\} \sim \{s_1, s_2\} \sim \{s_1, s_3\} \sim \{s_1, s_2, s_3\} \sim \emptyset$$

Note that the relation $\sqsupseteq$ is a total pre-order and $\{s_1\} \sqsupset A$ for all $A \in \mathcal{P}(S)$ with $A \neq \{s_1\}$. Let us see that $\sqsupseteq$ satisfies Property T. Let A, B, C, D be in $\mathcal{P}(S)$ such that $A \cap B = \emptyset$, $C \cap D = \emptyset$, $A \sqsupseteq B$ and $C \sqsupseteq D$. We want to show that $(A \cup C) \setminus (B \cup D) \sqsupseteq (B \cup D)$. Towards a contradiction, suppose $(A \cup C) \setminus (B \cup D) \not\sqsupseteq (B \cup D)$. Because $\sqsupseteq$ is total, $(B \cup D) \sqsupset (A \cup C) \setminus (B \cup D)$. By definition of the relation $\sqsupseteq$, necessarily $B \cup D = \{s_1\}$, thus $B = \{s_1\}$ or $D = \{s_1\}$. Without loss of generality, suppose $B = \{s_1\}$. Because $A \cap B = \emptyset$, we have $A \neq \{s_1\}$. Thus, $B \sqsupset A$ which is a contradiction. Therefore the relation $\sqsupseteq$ satisfies Property T.

Now put $A = \{s_1\}$, $B = \{s_2\} = C$ and $D = \{s_1, s_3\}$. Notice that $A \cap B = \emptyset$, $C \cap D = \emptyset$, $(A \cup C) \setminus (B \cup D) = \emptyset$, $A \sqsupset B$ and $C \sqsupseteq D$. Consider $x, y, z \in X$ such that $x >_X y >_X z$ and define $\succeq$ by DPR with $(\geq_X, \sqsupseteq)$. Then, by Lemma 7, $\succeq$ is not transitive.

We have just seen that the fact that $\sqsupseteq$ satisfies Property T is not enough in order to obtain the transitivity of the relation defined by DPR with $(\geq_X, \sqsupseteq)$. Thus, it is necessary to look for more properties in the relations over the events in order to have the relations over the policies (defined by DPR) be transitive. To this end we introduce the following property. It is a weakening of the monotony property (see for instance [3]).

***Weak monotony*(WM)** The relation $\sqsupseteq$ satisfies Weak Monotony if for all sets $A, B, C \in \mathcal{P}(S)$ the following properties hold:

SWM $A \supseteq B \sqsupseteq C \;\; \& \; A \cap C = \emptyset \Rightarrow A \sqsupseteq C$
IWM $A \sqsupseteq B \supseteq C \;\; \& \; A \cap B = \emptyset \Rightarrow A \sqsupseteq C$

It is not difficult to see that a possibilistic relation satisfies these properties. Another interesting example of a relation satisfying these properties is the leximin relation, $\sqsupseteq^L_{min}$. Actually, the relations having a probabilistic representation satisfy the property WM. We can prove that the relation $\sqsupseteq^L_{min}$ has a probabilistic representation. Thus, $\sqsupseteq^L_{min}$ and $\sqsupset^L_{min}$ satisfy the weak monotony property (WM).

Next we show that if the relation over the policies defined by DPR is transitive, then the relation over the events satisfies Weak Monotony. Notice, however, that the converse is not true because $\sqsupseteq^L_{min}$ is not transitive (Example 2) and, as we noted earlier, $\sqsupseteq^L_{min}$ satisfies Weak Monotony.

Proposition 9 *Let $\geq_X$ be a total pre-order over X such that there exist $x, y, z \in X$ such that $x >_X y >_X z$. Let $\sqsupseteq$ be a relation over $\mathcal{P}(S)$ which is consistent. Suppose that the relation $\succeq$ over the policies is defined by DPR with $(\geq_X, \sqsupseteq)$. If $\succeq$ is transitive, then $\sqsupseteq$ satisfies the property of Weak Monotony.*

Proof. First we will see that (SWM) holds: suppose $A \supseteq B \sqsupseteq C$ and $A \cap C = \emptyset$. Let f, g be in X^S such that $f = yBz$ and $g = yCz$. Since $B \cap C = \emptyset$, then $B = [f >_X g]$ and $C = [g >_X f]$. Thus, by DPR, $f \succeq g$. Consider $h(s) = x$ if $s \in A \setminus B$ and $h(s) = f(s)$ if $s \in (A \setminus B)^c$. Then $[h >_X f] = A \setminus B$ and $[f >_p h] = \emptyset$. Since $\sqsupseteq$ is consistent, by DPR, we have $h \succeq f$. But $\succeq$ is transitive, then $h \succeq g$. Now, $[h >_X g] = A$ and $[g >_X h] = C$, thus, by DPR, $A \sqsupseteq C$. Therefore SWM holds.
Now we check that IWM holds: suppose $A \sqsupseteq B \supseteq C$ and $A \cap B = \emptyset$. Let f, g in X^S such that $f = yAz$ and $g = yBz$. We have $A = [f >_X g]$ and $B = [g >_X f]$. By DPR $f \succeq g$. Taking $h(s) = g(s)$ if $s \in C \cup B^c$ and $h(s) = f(s)$ if $s \in B \setminus C$, we have $[g >_X h] = B \setminus C$ and $[h >_X g] = \emptyset$. By DPR, since $\sqsupseteq$ is consistent, we have $g \succeq h$. But $\succeq$ is transitive, then $f \succeq h$. Note that, $[f >_X h] = A$ and $[h >_X f] = C$, again, by DPR, $A \sqsupseteq C$. Therefore IWM holds.

We noted earlier that the relation $\sqsupseteq$ in Example 3 satisfies Property T. It is easy to check that this relation satisfies SWM and IWM. We have also seen that the relation over the policies in that Example is transitive. Actually, it is not an accidental fact. This is a general fact which we prove next, that is to say, the properties T and MD are enough for the transitivity to hold. More precisely, we have the following proposition:

Proposition 10 *Let $\succeq$ be a relation over X^S defined by DPR with $(\geq_X, \sqsupseteq)$. If $\sqsupseteq$ satisfies the properties T and WM, then $\succeq$ is transitive.*

Proof. Let f, g, h be in X^S such that $f \succeq g$ and $g \succeq h$. Consider $A = [f >_X g]$, $B = [g >_X f]$, $C = [g >_X h]$ and $D = [h >_X g]$. Notice $A \cap B = \emptyset$ and $C \cap D = \emptyset$. By DPR

$$A \sqsupseteq B \quad \text{y} \quad C \sqsupseteq D$$

and by Property T

$$(A \cup C) \setminus (B \cup D) \sqsupseteq B \cup D.$$

We claim $(A \cup C) \setminus (B \cup D) \subseteq [f >_X h]$ and $[h >_X f] \subseteq B \cup D$. Indeed,

$$\begin{aligned}(A \cup C) \setminus (B \cup D) &= (A \cap D^c) \cup (B^c \cap C)\\ &= ([f >_X g] \cap [g \geq_X h]) \cup ([f \geq_X g] \cap [g >_X h])\\ &= \{s \in S : f(s) >_X g(s) \geq_X h(s)\} \cup \{s \in S : f(s) \geq_X g(s) >_X h(s)\}\\ &\subseteq \{s \in S : f(s) >_X h(s)\}\\ &= [f >_X h],\end{aligned}$$

thus $(A \cup C) \setminus (B \cup D) \subseteq [f >_X h]$. For the second statement of our claim, consider $s \in [h >_X f]$, then $h(s) >_X f(s)$. Now consider $g(s)$. Because $\geq_X$ is total, only one of the following cases can hold:

1. $g(s) \geq_X h(s)$. Then, by the transitivity of $\geq_X$, $g(s) >_X f(s)$, thus $s \in B$;
2. $f(s) \geq_X g(s)$. Then, by the transitivity of $\geq_X$, $h(s) >_X g(s)$, thus $s \in D$;
3. $h(s) >_X g(s) >_X f(s)$, then $s \in B \cap D$,

Thus, in any case, $s \in B \cup D$.

We have $[f >_X h] \supseteq (A \cup C) \setminus (B \cup D)$, $(A \cup C) \setminus (B \cup D) \sqsupseteq B \cup D$ and $B \cup D \supseteq [h >_X f]$.
By IWM we have

$$(A \cup C) \setminus (B \cup D) \sqsupseteq [h >_X f]$$

and by SWM

$$[f >_X h] \sqsupseteq [h >_X f].$$

By DPR, $f \succeq h$, that is the relation $\succeq$ is transitive.

From Propositions 5, 9 and 10 we get the main result of this work: a representation theorem for the transitivity of the relation $\succeq$ defined by DPR with $(\geq_X, \sqsupseteq)$.

Theorem 11 *Let $\geq_X$ be a total pre-order over X such that there exist $x, y, z \in X$ satisfying $x >_X y >_X z$. Let $\sqsupseteq$ be a consistent relation over $\mathcal{P}(S)$. If $\succeq$ is defined by DPR with $(\geq_X, \sqsupseteq)$, then*

$$\succeq \textit{ is transitive} \quad \Leftrightarrow \quad \sqsupseteq \textit{ satisfies } T \textit{ and } WM$$

Next proposition tells us under which conditions Property T entails the Property of Weak Monotony.

Proposition 12 *Let $\sqsupseteq$ be a total relation satisfying Property T. If $\forall A \in S$, $(A \neq \emptyset \Rightarrow A \sqsupset \emptyset)$, then $\sqsupseteq$ satisfies the properties of Weak Monotony.*

Proof. First we will see that Property SWM holds. Suppose $A \supseteq B$, $B \sqsupseteq C$ and $A \cap C = \emptyset$. We want to show $A \sqsupseteq C$. If $C = \emptyset$, by hypothesis, $A \sqsupseteq C$. If $C \neq \emptyset$, because $B \sqsupseteq C$, then $B \neq \emptyset$, thus $A \neq \emptyset$. We argue by reductio, suppose $\neg(A \sqsupseteq C)$. Since $\sqsupseteq$ is total, $C \sqsupseteq A$. Since $A \cap C = \emptyset$ and $B \cap C = \emptyset$, by Property T, $(B \cup C) \setminus (A \cup C) \sqsupseteq A \cup C$. But $(B \cup C) \setminus (A \cup C) = \emptyset$ because $A \supseteq B$. Therefore $\emptyset \sqsupseteq A \cup C$, which is a contradiction since $A \cup C \neq \emptyset$ and by hipothesis $A \cup C \sqsupset \emptyset$. Therefore $A \sqsupseteq C$.
Now we will see that Property IWM is satisfied. Suppose $A \sqsupseteq B$, $B \supseteq C$ and $A \cap B = \emptyset$. We want to show $A \sqsupseteq C$. If $C = \emptyset$, by hypothesis, $A \sqsupseteq C$. If $C \neq \emptyset$, then $B \neq \emptyset$ and because $\neg(\emptyset \sqsupseteq B)$, we have $A \neq \emptyset$. Towards a contradiction, suppose $\neg(A \sqsupseteq C)$. Since $\sqsupseteq$ is total, $C \sqsupseteq A$. But we know $A \cap C = \emptyset$. Using Property T,

$$(A \cup C) \setminus (A \cup B) \sqsupseteq A \cup B.$$

But

$(A \cup C) \setminus (A \cup B) = \emptyset$ since $B \supseteq C$, therefore

$$\emptyset \sqsupseteq A \cup B$$

which is a contradiction because $A \cup B \neq \emptyset$ and, by hypothesis $A \cup B \sqsupset \emptyset$. Thus, $A \sqsupseteq C$.

From Theorem 11 and Proposition 12 we get straightforward the following corollary:

Corollary 13 *Let $\geq_X$ be a total pre-order over X such that there exist $x, y, z \in X$ satisfying $x >_X y >_X z$. Let $\sqsupseteq$ be a total relation over $\mathcal{P}(S)$ such that $\forall A \in S, (A \neq \emptyset \Rightarrow A \sqsupset \emptyset)$. If $\succeq$ is defined by DPR with $(\geq_X, \sqsupseteq)$, then*

$$\succeq \textit{ is transitive} \Leftrightarrow \quad \sqsupseteq \textit{ satisfies Property} \quad T$$

Notice also the following corollary from Theorem 11:

Corollary 14 *Let $\succeq$ be defined by DPR with $(\geq_X, \sqsupseteq_\Pi)$, where $\sqsupseteq_\Pi$ is possibilistic. Then the strict relation $\succ$, associated to $\succeq$, is transitive.*

Proof. This is due to the following two observations the proofs of which are quite straightforward:

- $f \succ g$ iff $[f >_X g] \sqsupset_\Pi [g >_X f]$.
- The relation $\sqsupseteq_\Pi$ satisfies the properties T and MD.

Then by Theorem 11 we conclude that $\succ$ es transitive.

4 Concluding remarks

In this work we have found a very general characterization of the transitivity for the relation over the policies defined by the Dominance Plausible Rule.

We have seen some applications of the characterization. Namely, the relations over the policies defined by DPR via a possibilistic relation or a leximin are not transitive. On the other hand if we use a strict possibilistic relation, the relation over the policies obtained via DPR is transitive.

The given characterization put on evidence tight links between the transitivity of the relation over the policies and the monotony of the relation over the events used in the definition.

Acknowledgements

Thanks to LOFT2010's referees for their careful reading and their useful suggestions.

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