COMP329 Robotics and Autonomous Systems

Lecture 12: Kinematics

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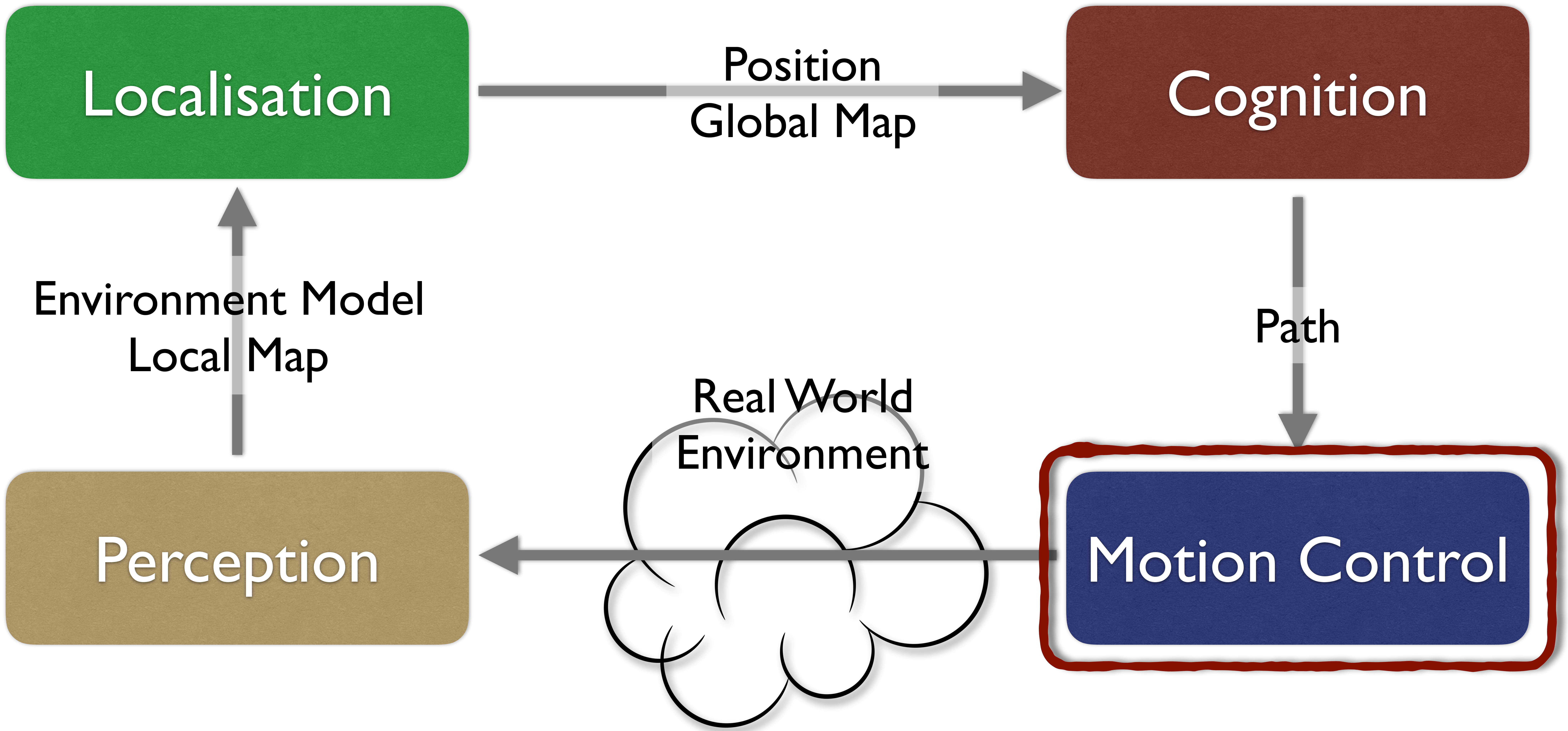
General control architecture



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Locomotion & Kinematics

- Two aspects to motion:
  - Locomotion
  - Kinematics
- *Locomotion*:
  - What kinds of motion are possible?
  - What physical structures are there?
- ***Kinematics***:
  - Mathematical model of motion.
  - Models make it possible to predict motion.

Kinematics

- So far we have looked at different kinds of motion in a qualitative way.
  - One way to program robots to move is ***trial and error***.
  - A somewhat better way is to establish mathematically how the robot ***should*** move, this is ***kinematics***.
- Rather *kinematics* is the business of figuring how a robot will move if its motors work in a given way.
- *Inverse-kinematics* then tells us how to move the motors to get the robot to do what we want.
- We'll look at a few tiny bits of the kinematics world.

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Formal Model

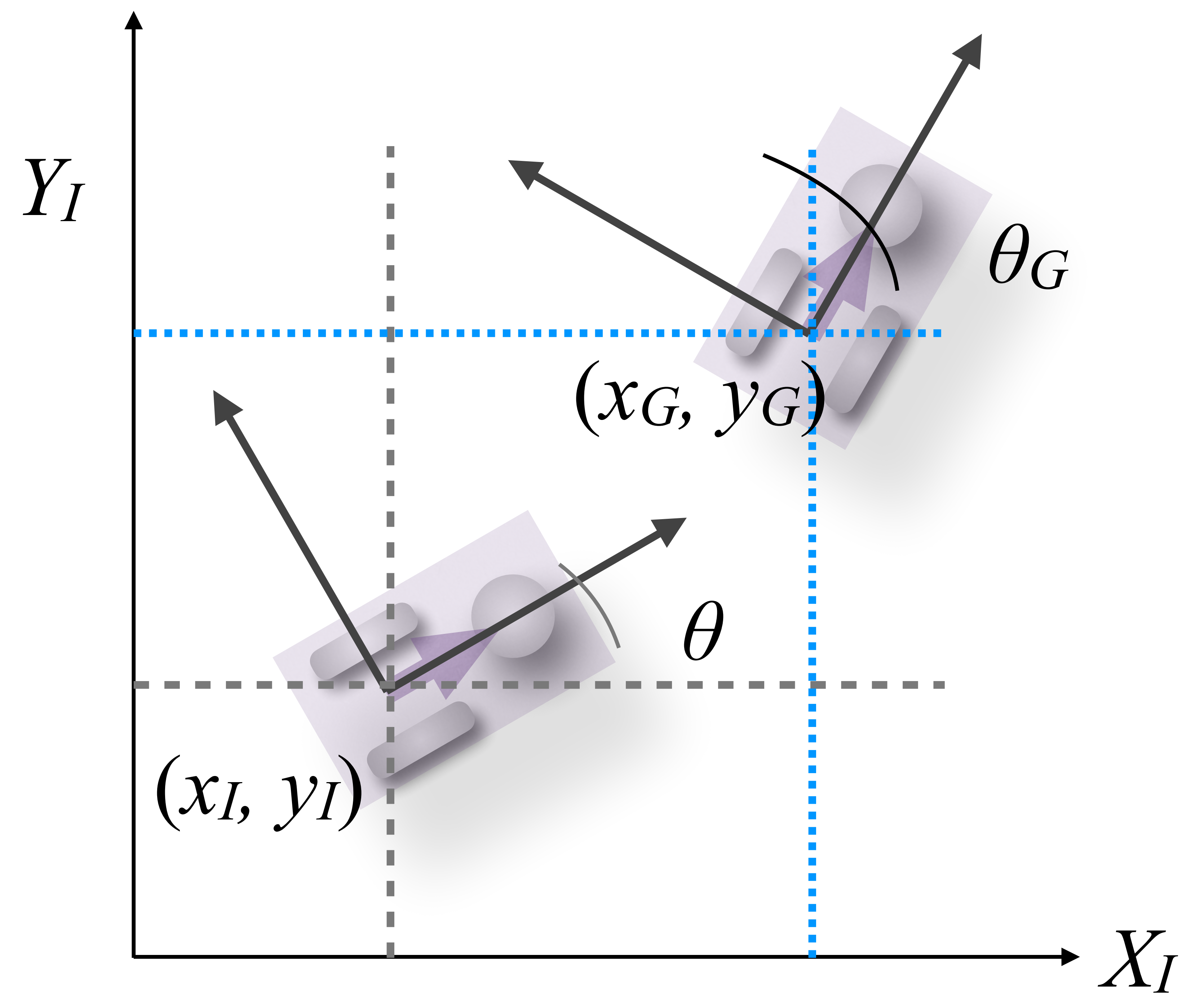
- We will assume, as people usually do, that the robot's location, or *pose* is fixed in terms of three coordinates:

$$(x_I, y_I, \theta)$$

- Given that the robot needs to navigate to a new location:

$$(x_G, y_G, \theta_G)$$

- …it can determine how x, y and θ need to change.
  - BUT it can't control these directly.



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Kinematic Model

- All a robot has access to are the speeds of its wheels:

$$\dot{\varphi}_1, \ldots \dot{\varphi}_n$$

- The steering angle of the steerable wheels:

$$\beta_1, \ldots \beta_m$$

- And the speed with which those steering angles are changing.

$$\dot{\beta}_1, \ldots \dot{\beta}_m$$

- Together these determine the motion of the robot:

$$f(\dot{\varphi}_1, \ldots \dot{\varphi}_n, \beta_1, \ldots \beta_m, \dot{\beta}_1, \ldots \dot{\beta}_m) = \begin{bmatrix} \dot{x}_I \\ \dot{y}_I \\ \dot{\theta} \end{bmatrix}$$

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Reverse Kinematics

- For Reverse Kinematics, this model is not what we want:

$$f(\dot{\varphi}_1, \ldots \dot{\varphi}_n, \beta_1, \ldots \beta_m, \dot{\beta}_1, \ldots \dot{\beta}_m) = \begin{bmatrix} \dot{x}_I \\ \dot{y}_I \\ \dot{\theta} \end{bmatrix}$$

- We want to know how to set $\dot{\varphi}_i$ etc to get a given:

$$\dot{x}_I, \dot{y}_I, \dot{\theta}$$

- We can get what we want from the forward kinematic model:

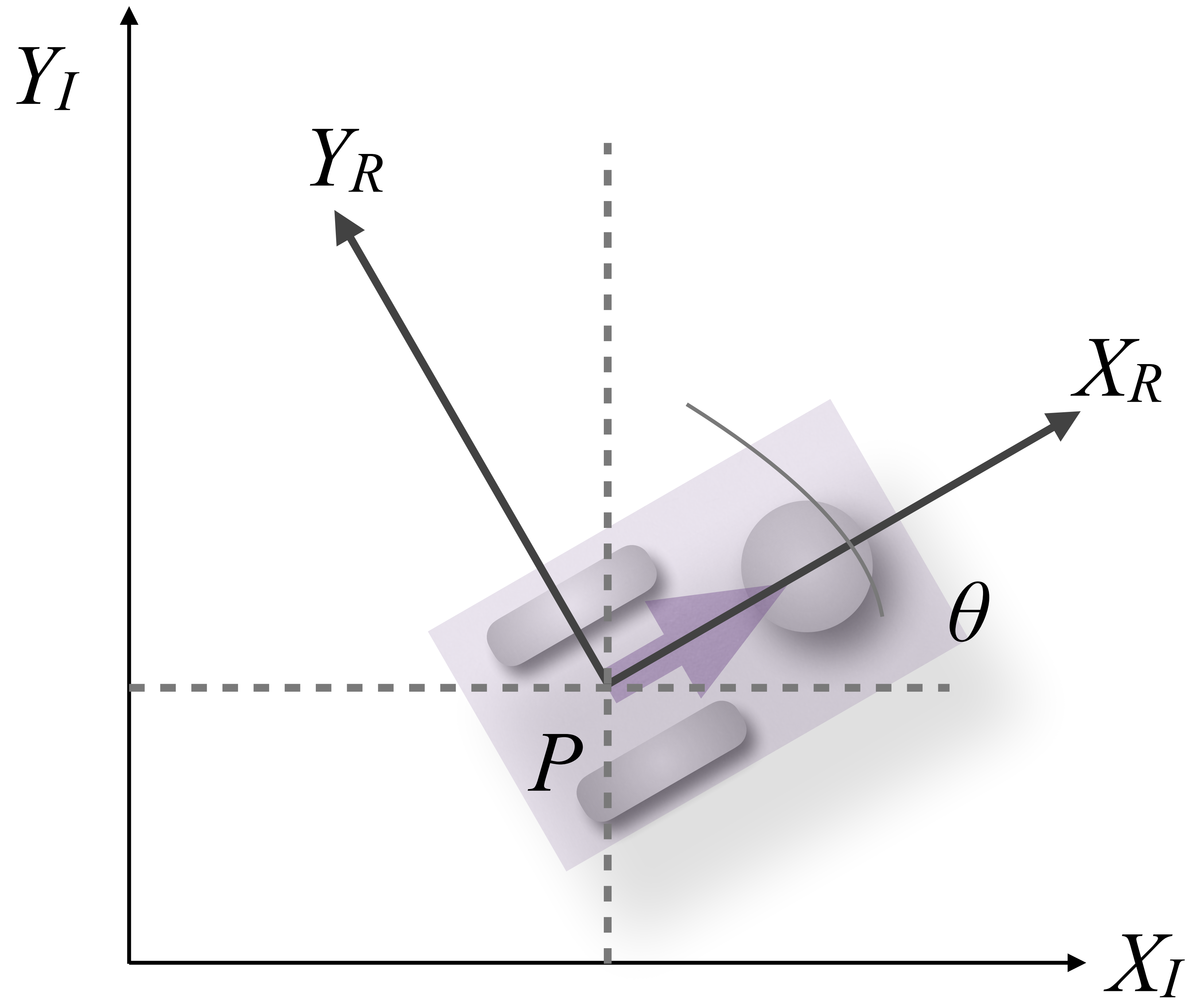
$$\begin{bmatrix} \dot{\varphi}_1 \\ \vdots \\ \dot{\varphi}_n \\ \beta_1 \\ \vdots \\ \beta_m \\ \dot{\beta}_1 \\ \vdots \\ \dot{\beta}_m \end{bmatrix} = f^{-1}(\dot{x}_I, \dot{y}_I, \dot{\theta})$$

Three Problems in Kinematics

1. Transformation between frames.
2. Reversing the kinematic model.
3. Deriving robot motion from robot structure.

Representing the robot's position

- The robot knows how it moves relative to its centre of rotation.
- This is not the same as knowing how it moves relative to the world



Two systems of coordinates:

Initial Frame: $\{X_I, Y_I\}$
Robot Frame: $\{X_R, Y_R\}$

Where X_R is rotated from X_I by θ radians

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Frame Transformation

- Robot Position:

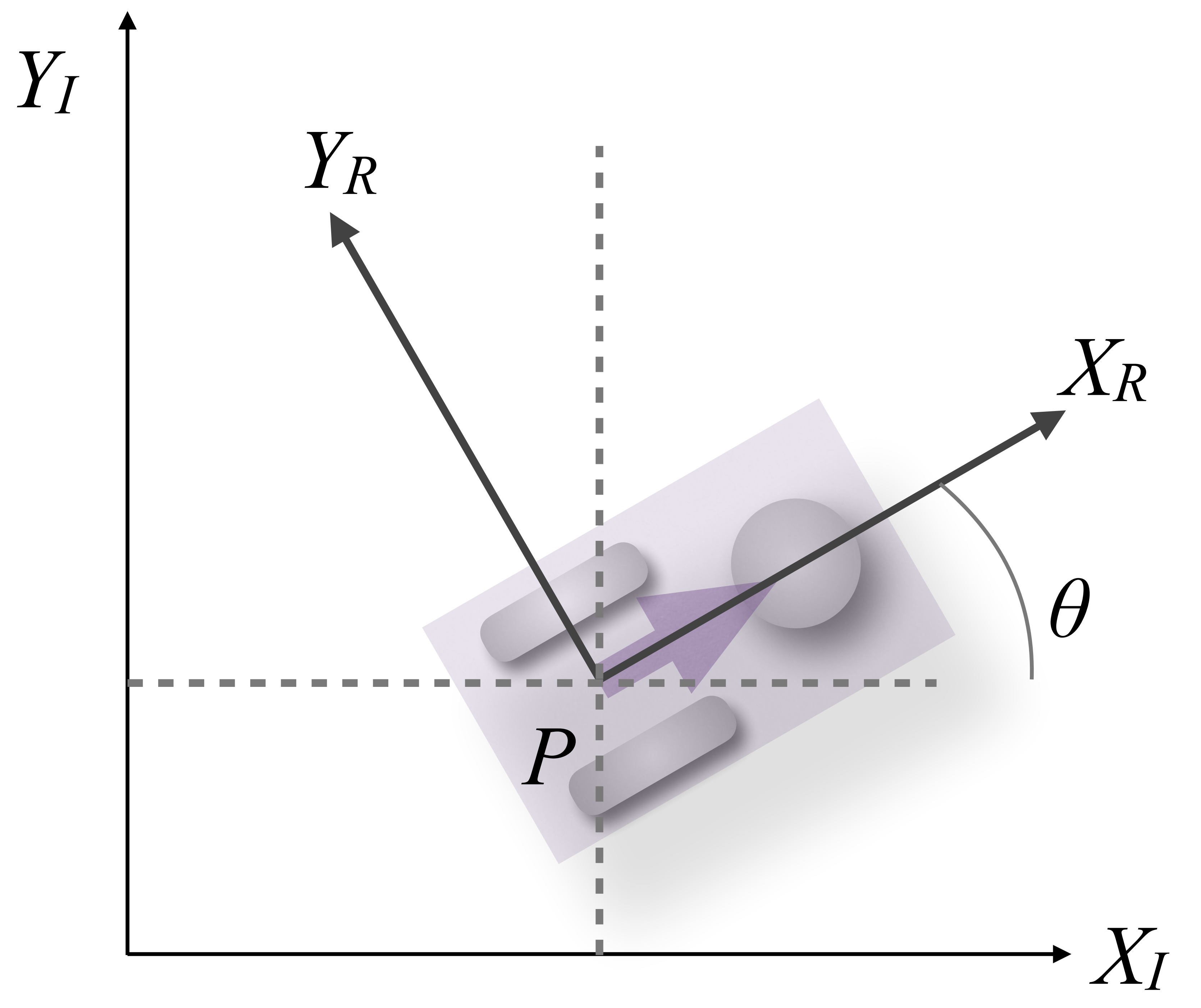
$$\xi_I = [x_I, y_I, \theta_I]^T$$

- Mapping between frames:

$$\begin{aligned}\dot{\xi_R} &= R(\theta)\dot{\xi_I} \\ &= R(\theta)\left[\dot{x_I}, \dot{y_I}, \dot{\theta_I}\right]^T\end{aligned}$$

where

$$R(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



- $R(\theta)$ is the ***Orthogonal Rotation Matrix***

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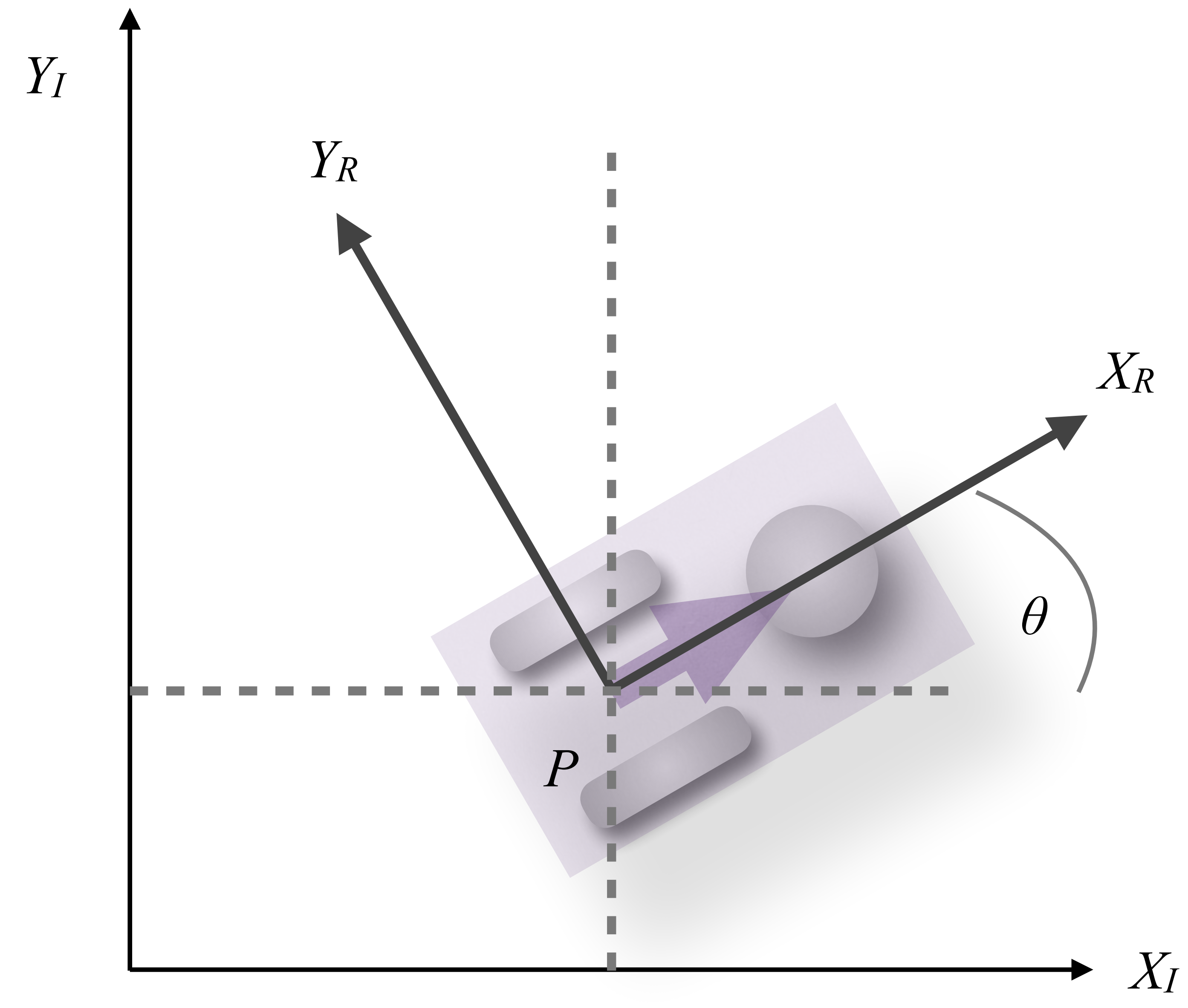
Frame Transformation

- In other words

$$\begin{bmatrix} \dot{x}_R \\ \dot{y}_R \\ \dot{\theta}_R \end{bmatrix} = R(\theta) \begin{bmatrix} \dot{x}_I \\ \dot{y}_I \\ \dot{\theta}_I \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_I \\ \dot{y}_I \\ \dot{\theta}_I \end{bmatrix}$$

Orthogonal Rotation Matrix

$$R(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



meaning that

$$\begin{aligned} \dot{x}_R &= \dot{x}_I \cos(\theta) + \dot{y}_I \sin(\theta) + \dot{\theta}_I.0 \\ \dot{y}_R &= -\dot{x}_I \sin(\theta) + \dot{y}_I \cos(\theta) + \dot{\theta}_I.0 \\ \dot{\theta}_R &= \dot{x}_I.0 + \dot{y}_I.0 + \dot{\theta}_I.1 = \dot{\theta}_I \end{aligned}$$

Frame Transformation

- In other words

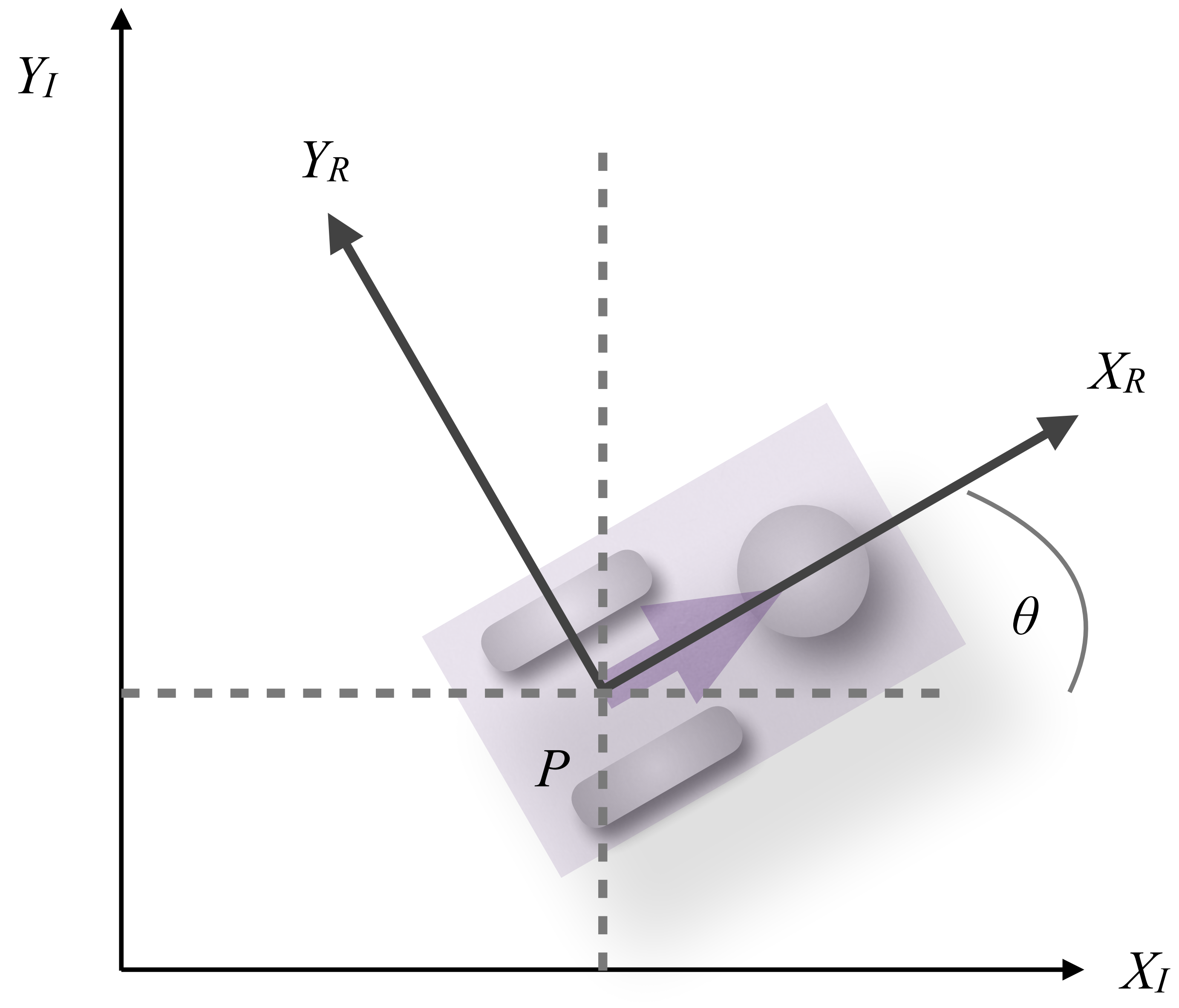
$$\begin{bmatrix} \dot{x}_R \\ \dot{y}_R \\ \dot{\theta}_R \end{bmatrix} = R(\theta) \begin{bmatrix} \dot{x}_I \\ \dot{y}_I \\ \dot{\theta}_I \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_I \\ \dot{y}_I \\ \dot{\theta}_I \end{bmatrix}$$

Orthogonal Rotation Matrix

$$R(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

meaning that

$$\begin{aligned} \dot{x}_R &= \dot{x}_I \cos(\theta) + \dot{y}_I \sin(\theta) \\ \dot{y}_R &= \dot{y}_I \cos(\theta) - \dot{x}_I \sin(\theta) \\ \dot{\theta}_R &= \dot{\theta}_I \end{aligned}$$



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Frame Transformation

- In other words, given how the robot moves in the world, we can calculate ***how the robot moves relative to its centre of rotation***.
  - This is (part of) the forward kinematic model
  - But this isn't what we want!!!

- We want to be able to calculate ***how the robot moves in the world***, given how it moves relative to its centre of rotation.
  - That is, we want the *reverse* of this model.

Reverse Kinematics

- We want the ***reverse kinematic*** model:

$$\begin{bmatrix} \dot{x}_I \\ \dot{y}_I \\ \dot{\theta}_I \end{bmatrix} = R(\theta)^{-1} \begin{bmatrix} \dot{x}_R \\ \dot{y}_R \\ \dot{\theta}_R \end{bmatrix}$$

  where $R(\theta)^{-1}$ is the inverse of $R(\theta)$.

- Often $R(\theta)^{-1}$ is hard to compute, but luckily for us ***in this case*** it isn't.

- We have:

$$R(\theta)^{-1} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

  which we can use to establish $\dot{x}_I, \dot{y}_I, \dot{\theta}_I$

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Reverse Kinematics

- To do this we compute:

$$\begin{bmatrix} \dot{x}_I \\ \dot{y}_I \\ \dot{\theta}_I \end{bmatrix} = R(\theta)^{-1} \begin{bmatrix} \dot{x}_R \\ \dot{y}_R \\ \dot{\theta}_R \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_R \\ \dot{y}_R \\ \dot{\theta}_R \end{bmatrix}$$

Inverse Orthogonal Rotation Matrix

$$R(\theta)^{-1} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

meaning that:

$$\begin{aligned} \dot{x}_I &= \dot{x}_R \cos(\theta) - \dot{y}_R \sin(\theta) + \dot{\theta}_R.0 \\ \dot{y}_I &= \dot{x}_R \sin(\theta) + \dot{y}_R \cos(\theta) + \dot{\theta}_R.0 \\ \dot{\theta}_I &= \dot{x}_R.0 + \dot{y}_R.0 + \dot{\theta}_R.1 \end{aligned}$$

Reverse Kinematics

- To do this we compute:

$$\begin{bmatrix} \dot{x}_I \\ \dot{y}_I \\ \dot{\theta}_I \end{bmatrix} = R(\theta)^{-1} \begin{bmatrix} \dot{x}_R \\ \dot{y}_R \\ \dot{\theta}_R \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_R \\ \dot{y}_R \\ \dot{\theta}_R \end{bmatrix}$$

meaning that:

$$\begin{aligned} \dot{x}_I &= \dot{x}_R \cos(\theta) - \dot{y}_R \sin(\theta) \\ \dot{y}_I &= \dot{x}_R \sin(\theta) + \dot{y}_R \cos(\theta) \\ \dot{\theta}_I &= \dot{\theta}_R. \end{aligned}$$

Inverse Orthogonal Rotation Matrix

$$R(\theta)^{-1} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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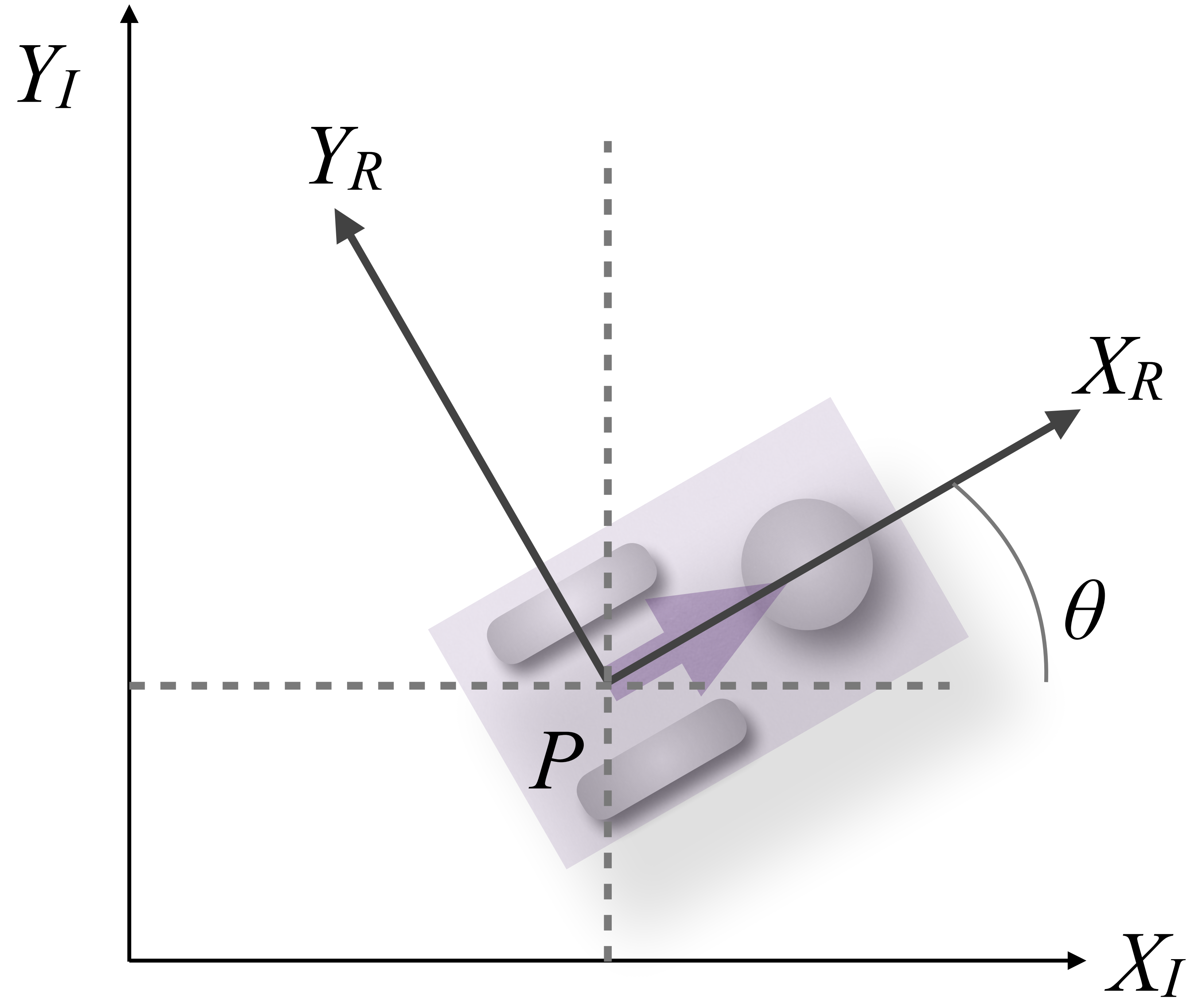
Down to the structure of the robot

- We can now identify the motion of the robot, in the global frame, if we know:

$$\dot{x}_R, \dot{y}_R, \dot{\theta}$$

  - but how do we tell what these are?



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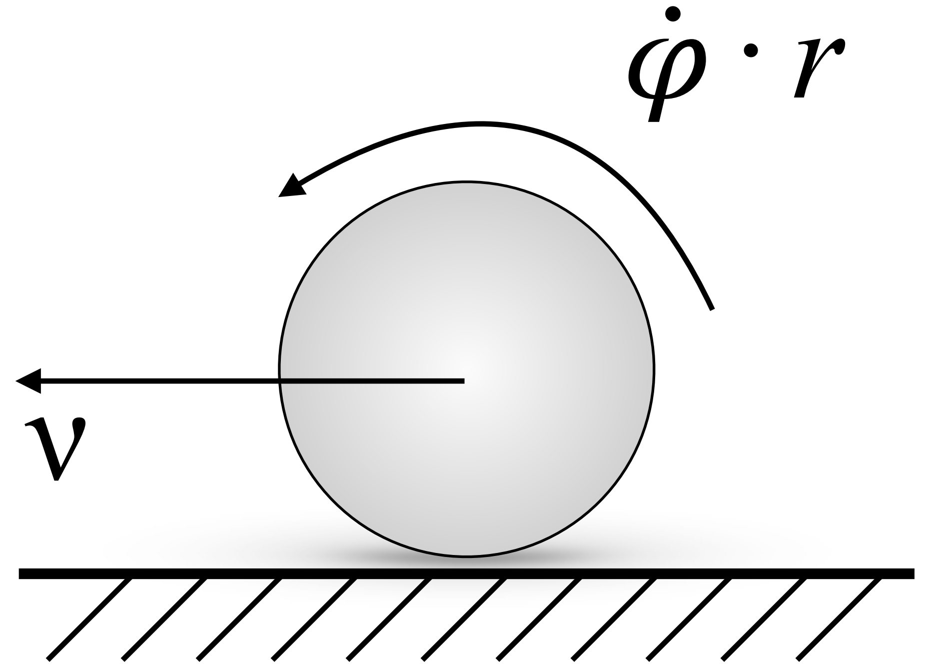
Down to the structure of the robot

- We compute them from what we can measure, like the speed of the wheels.
- Some assumptions (*constraints*) on the motion of the robot:
  - Movement on a horizontal plane
  - Point contact of the wheels; wheels not deformable
  - Pure rolling, so v=0 at contact point; no slipping, skidding or sliding
  - No friction for rotation around contact point
  - Steering axes orthogonal to the surface
  - Wheels connected by rigid frame (chassis)



- These won't always be true, why?

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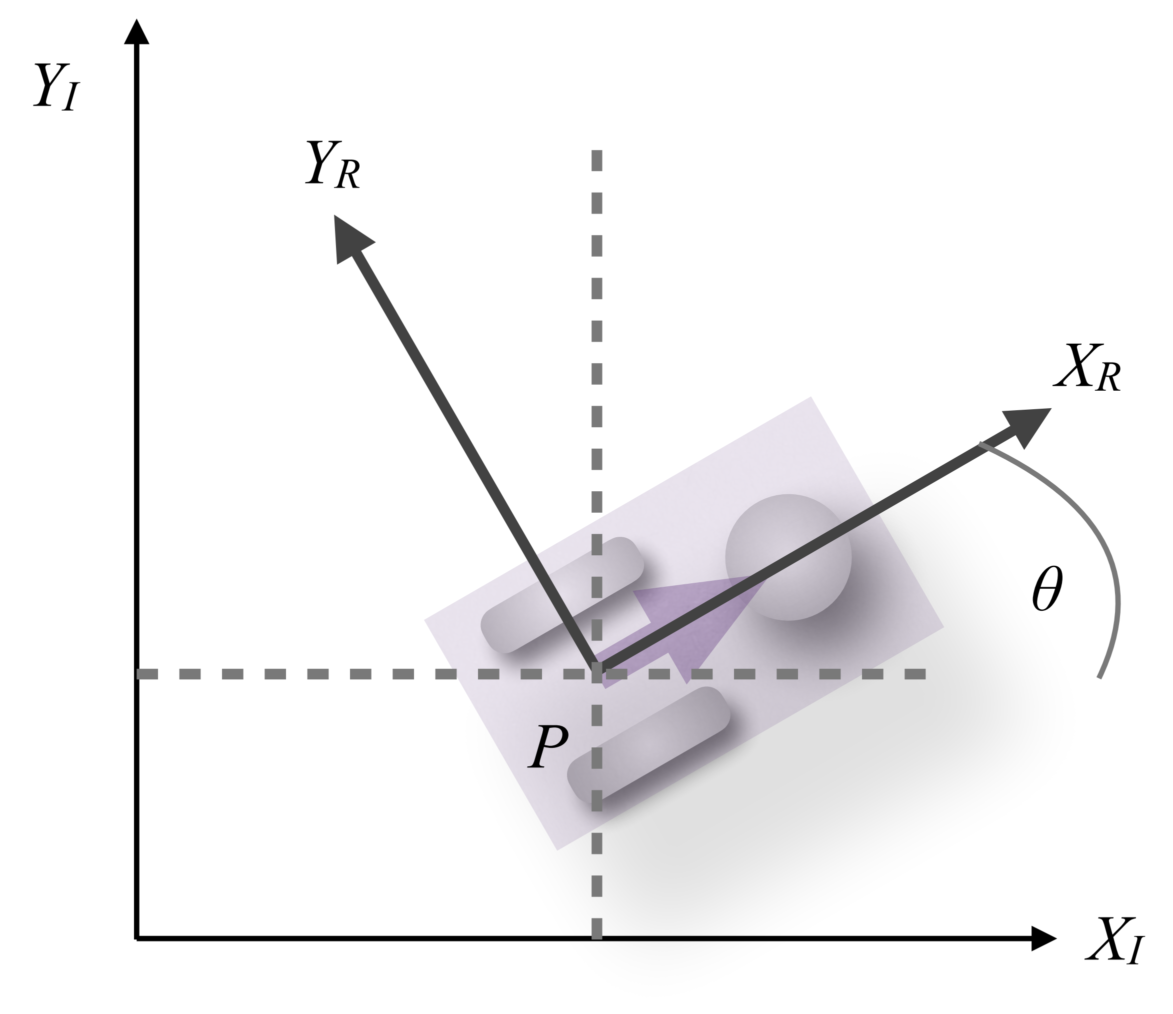
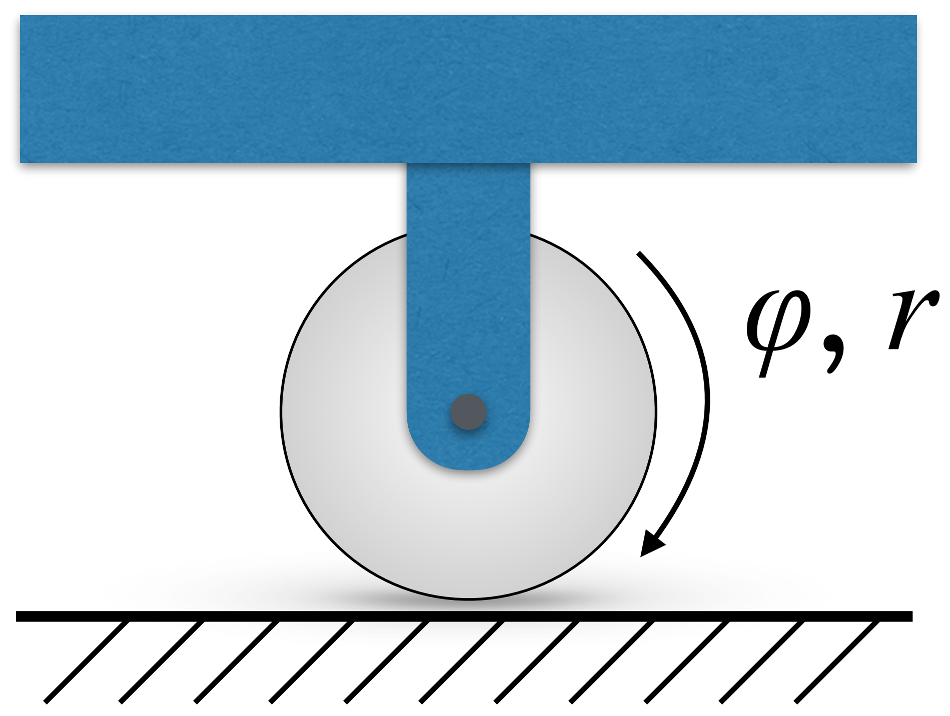
Differential Drive

- Consider differential drive.
  - Wheels of diameter r rotate at $\dot{\varphi}$ radians per second
    - Left wheel: $\dot{\varphi}_1$
    - Right wheel: $\dot{\varphi}_2$
  - Each wheel contributes: $\frac{r\dot{\varphi}}{2}$

    to motion of centre of rotation.





- Motion in the x direction.
  - Total speed is the sum of two contributions.

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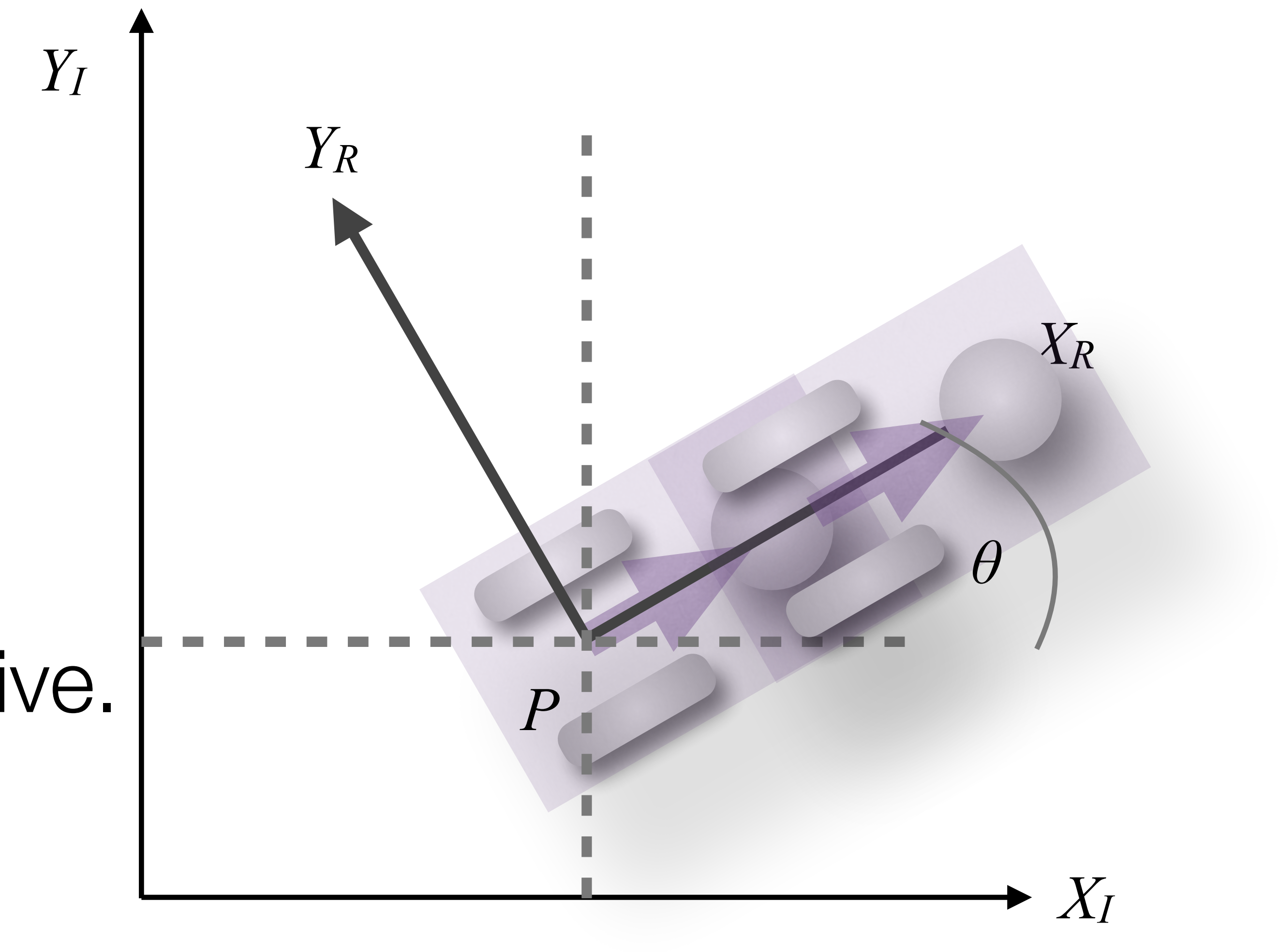
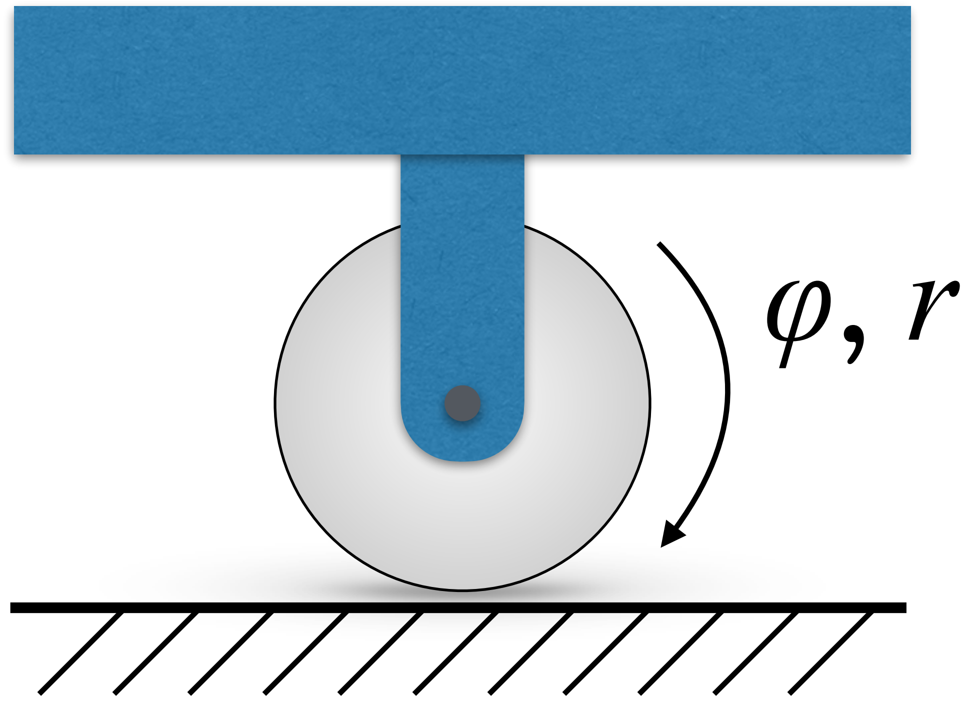
Differential Drive

- Example 1:
  - Assume each wheel has a diameter $r=1$
  - Each wheel will move a full rotation
    - i.e. will move 2π radians
  - As $\dot{\varphi}_1 = \dot{\varphi}_2$, the robot will move only along the x axis (i.e. forward)
  - This is because there is no lateral movement (i.e. along the y axis) differential drive.
  - Wheels of diameter r rotate at $\dot{\varphi}$ radians per second
    - Left wheel: $\dot{\varphi}_1$
    - Right wheel: $\dot{\varphi}_2$
  - The Robot centre (P) moves:

$$\begin{aligned} x &= \frac{r\dot{\varphi}_1}{2} + \frac{r\dot{\varphi}_2}{2} \\ &= \frac{1 \times 2\pi}{2} + \frac{1 \times 2\pi}{2} \\ &= 2\pi \end{aligned}$$

  - i.e. the circumference of the wheel!





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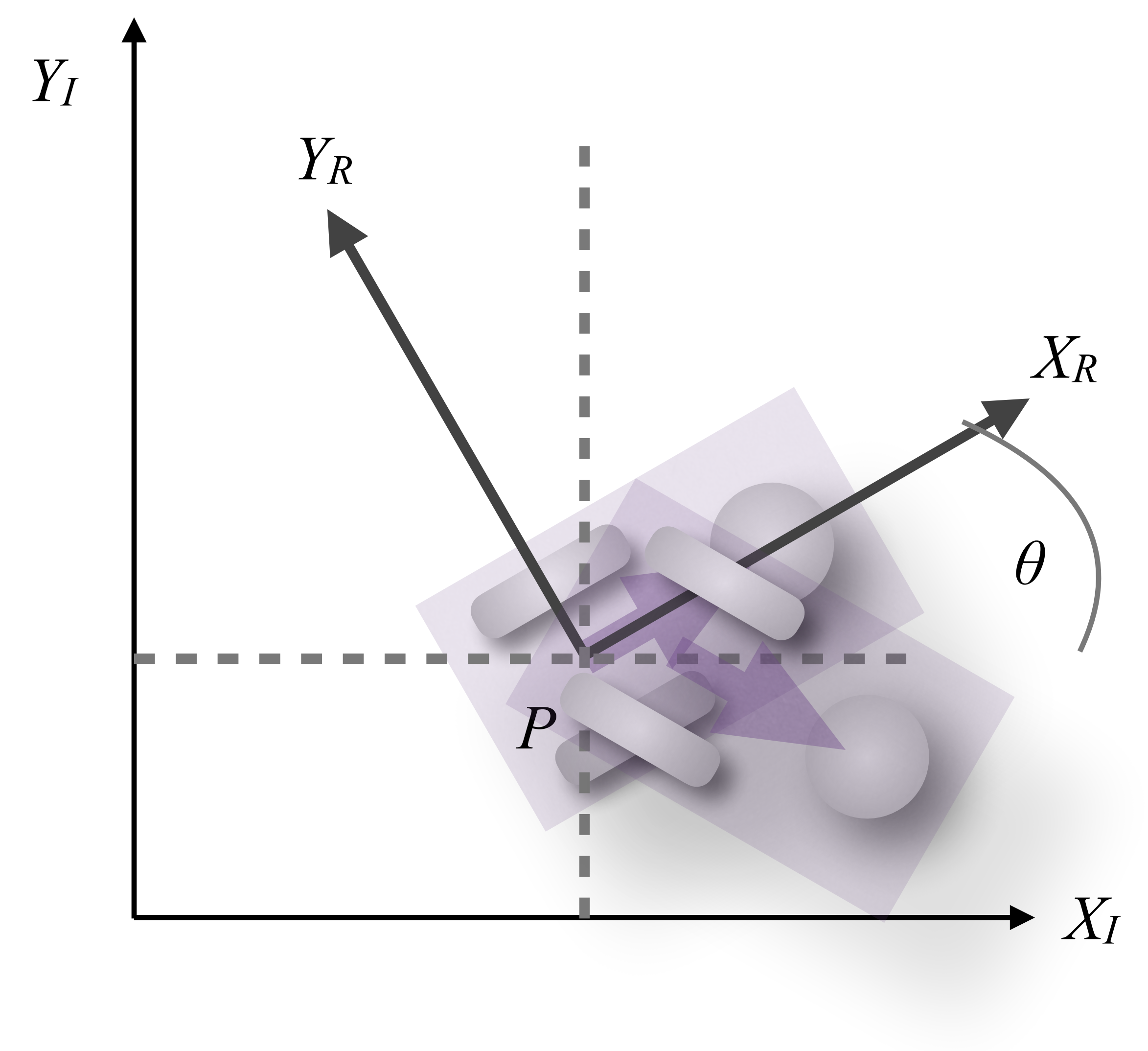
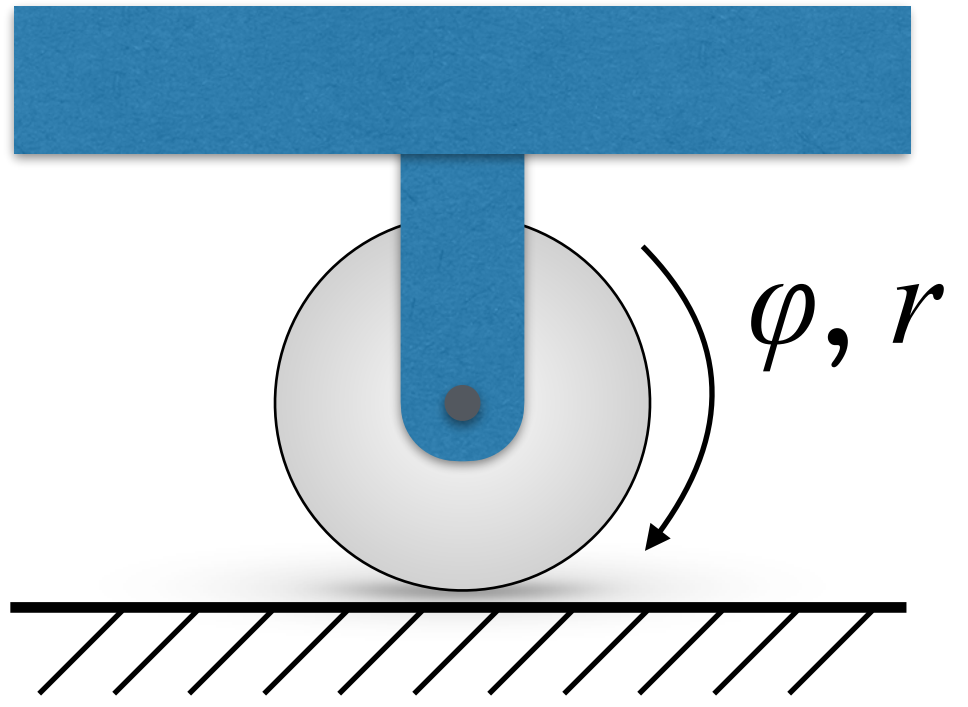
Differential Drive

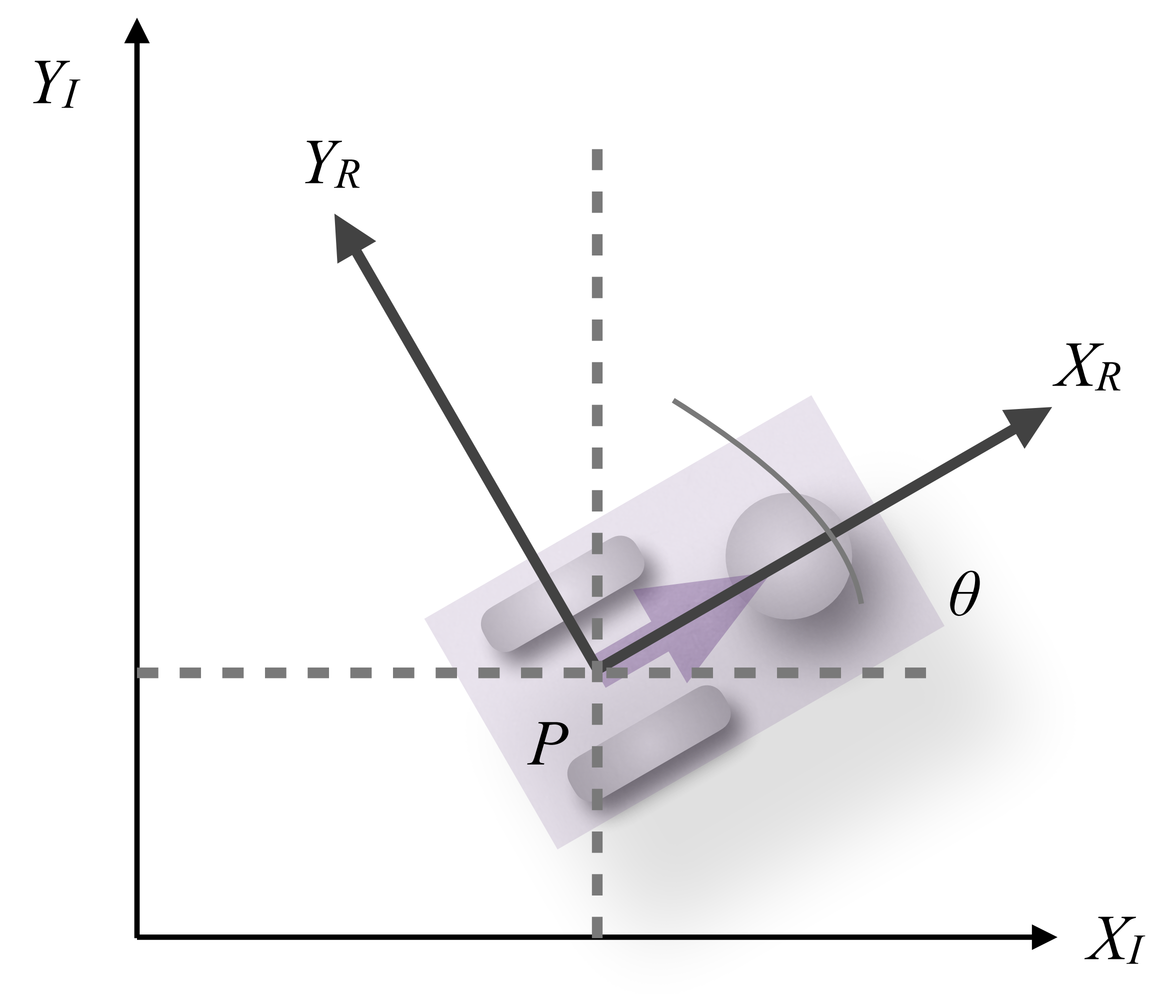
- Example 2:
  - Assume each wheel has a diameter $r=1$
  - Only the left wheel moves a full rotation
    - Right wheel is stationary
  - The robot will now move around the right wheel
  - Centre of the Robot moves: $$x = \frac{r\dot{\varphi}_1}{2} + \frac{r\dot{\varphi}_2}{2} = \frac{1 \times 2\pi}{2} + 0 = \pi$$
  - However, we have not calculated the change θ in angle or movement in the y axis



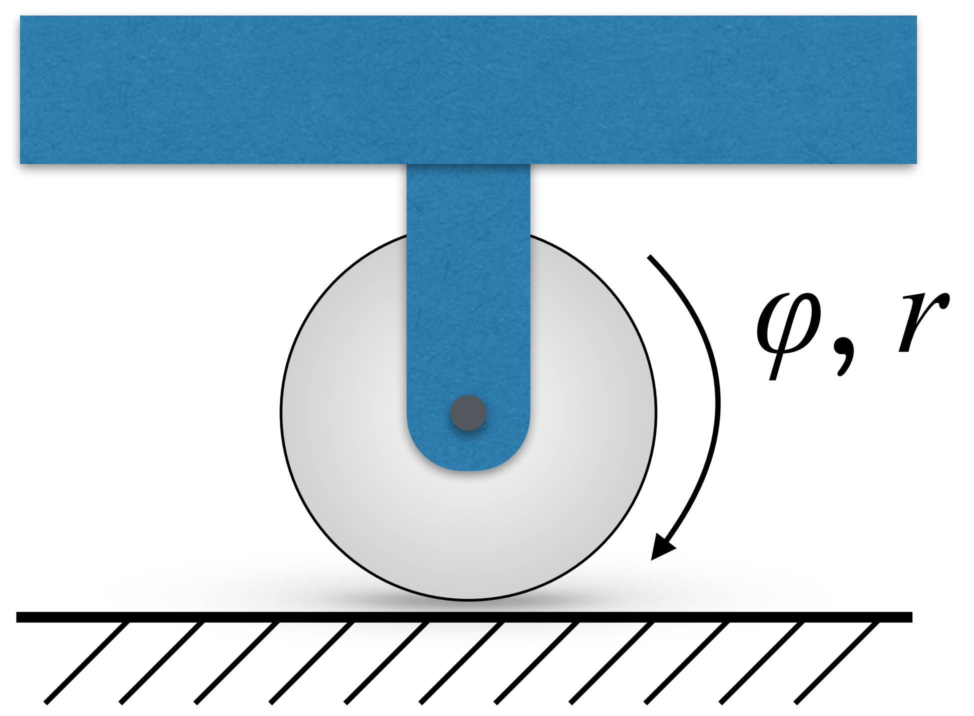


Differential Drive

- What if wheels move in counter directions?
  - Now, motion in the θ direction.



- Rotation due to ***left*** wheel (going forward) is: $\omega_1 = \frac{r\dot{\varphi}_1}{2l}$
  - l is the distance from P to a wheel.



- Whereas rotation due to the right wheel (going backward) is: $\omega_2 = \frac{-r\dot{\varphi}_2}{2l}$

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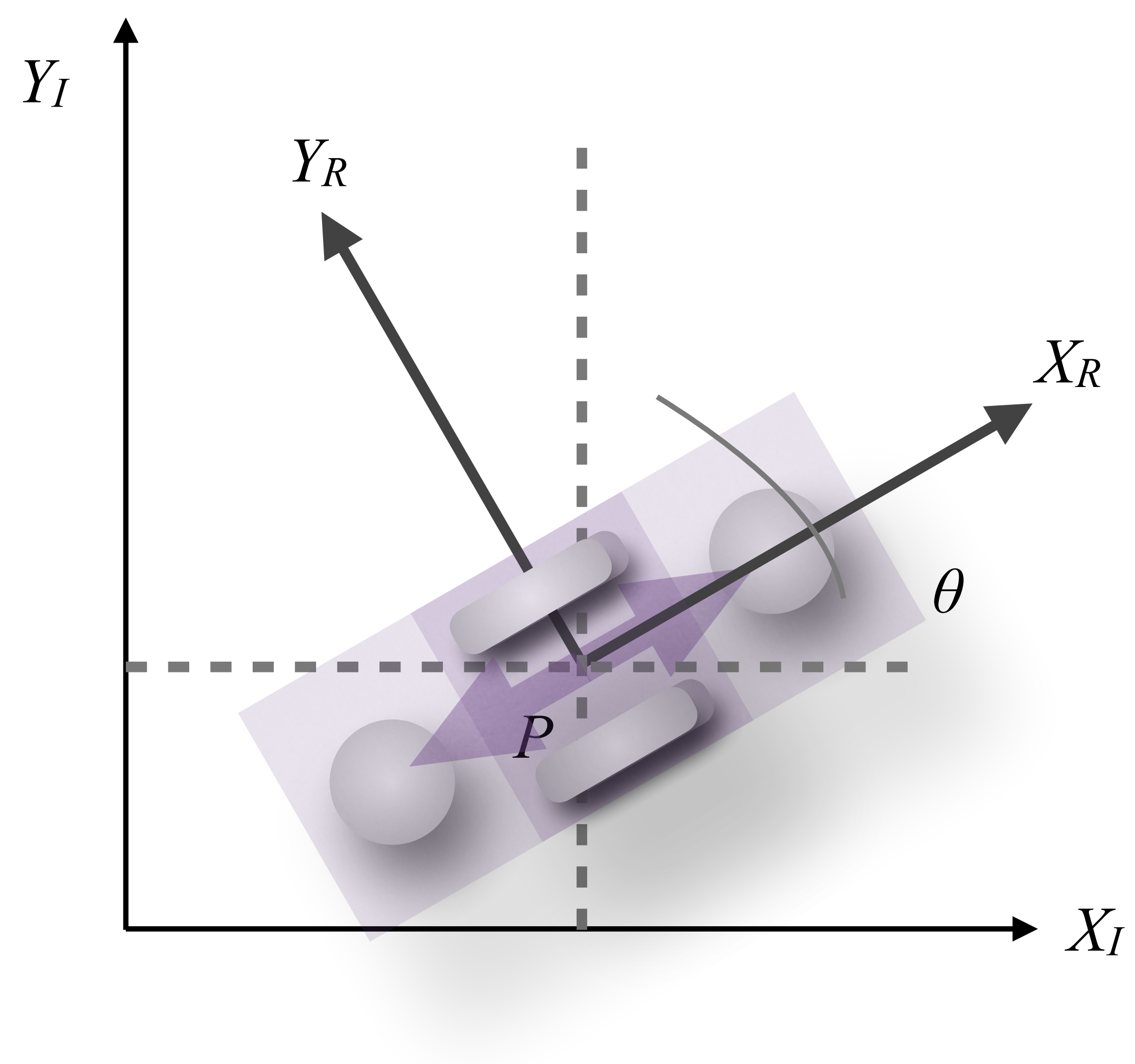
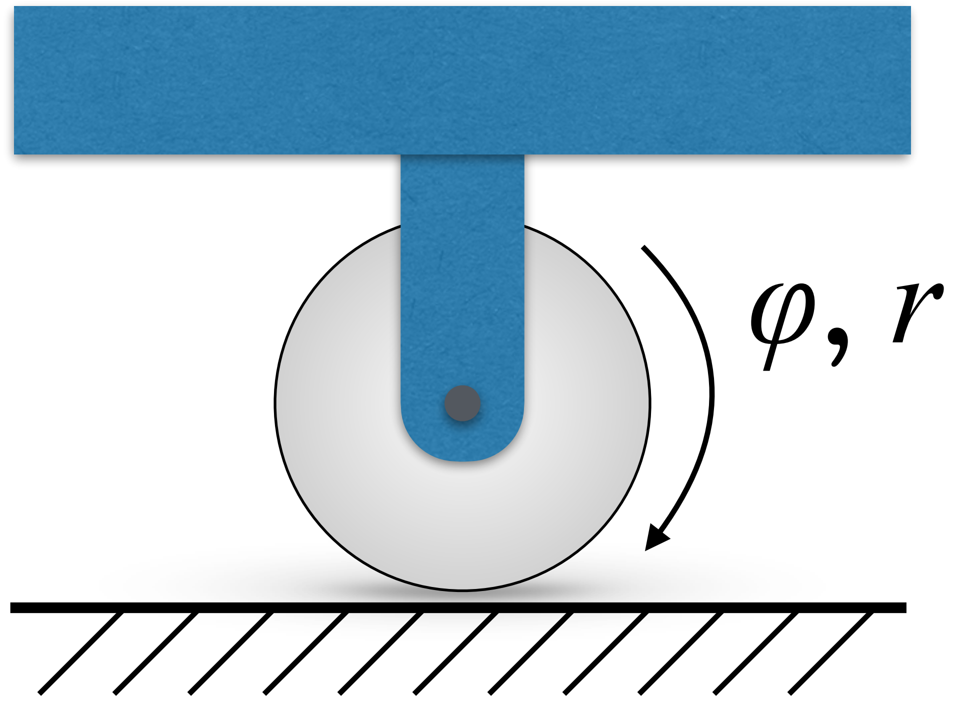
Differential Drive

- Combining these components we have:

$$\dot{\xi}_I = \begin{bmatrix} \dot{x}_R \\ \dot{y}_R \\ \dot{\theta}_R \end{bmatrix} = \begin{bmatrix} \frac{r\dot{\varphi}_1}{2} + \frac{r\dot{\varphi}_2}{2} \\ 0 \\ \frac{r\dot{\varphi}_1}{2l} - \frac{r\dot{\varphi}_2}{2l} \end{bmatrix}$$

- And we can combine these with $R(\theta)^{-1}$ to find motion in the global frame.





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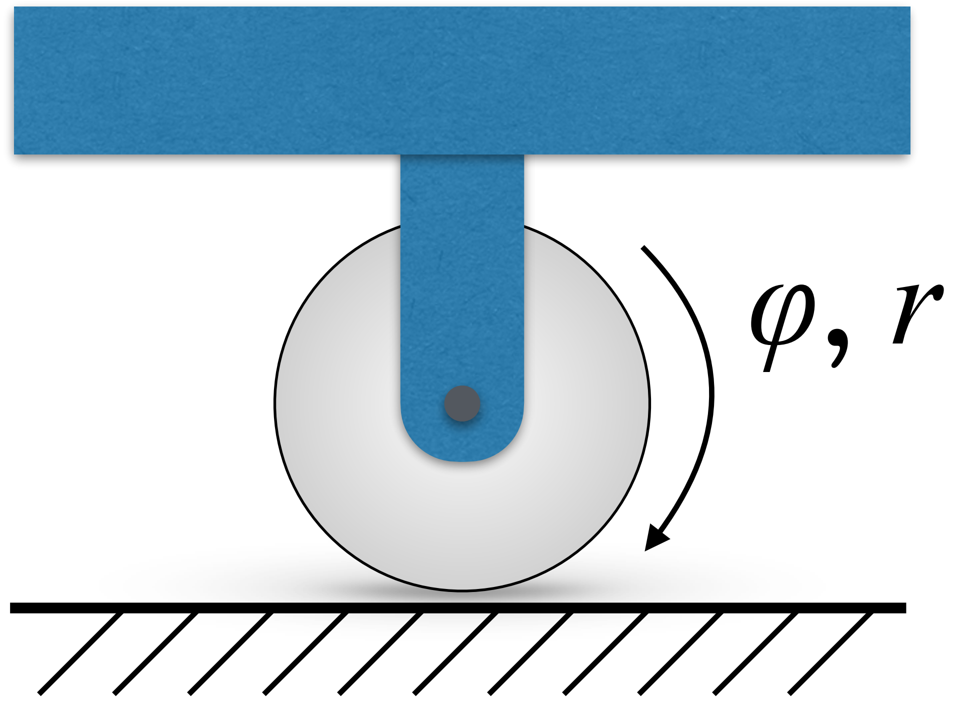
Differential Drive

- Suppose that the robot is positioned such that:
  - $\theta = \pi/2$
  - $r = l = 1$

- If the robot engages its wheels ***unevenly***, such that:
  - $\omega_1 = 4$
  - $\omega_2 = 3$



We can compute its velocity in the global reference frame

$$\dot{\xi}_R = \begin{bmatrix} \frac{r\dot{\varphi}_1}{2} + \frac{r\dot{\varphi}_2}{2} \\ 0 \\ \frac{r\dot{\varphi}_1}{2l} - \frac{r\dot{\varphi}_2}{2l} \end{bmatrix} = \begin{bmatrix} \frac{1\times4}{2} + \frac{1\times2}{2} \\ 0 \\ \frac{1\times4}{2\times1} - \frac{1\times2}{2\times1} \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

$$\dot{\xi}_I = R(\theta)^{-1}\dot{\xi}_R = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$$

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Differential Drive

- Suppose that the robot is positioned such that:
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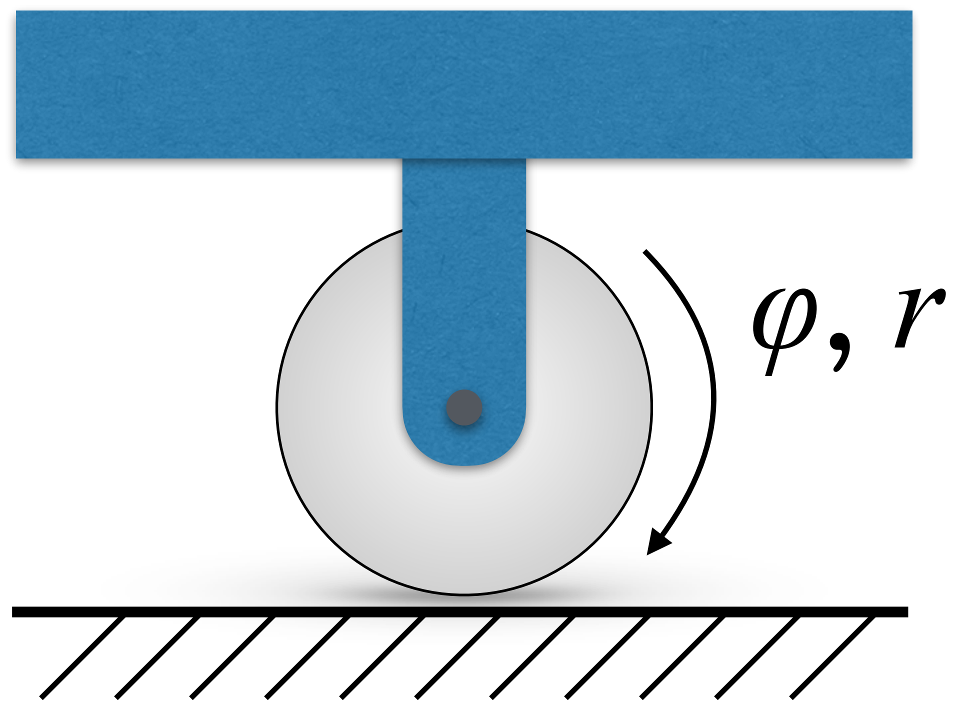

Thus, the robot will move:

Along the y axis of the global reference frame

Speed 3 / Rotating speed 1

$$\dot{\xi}_I = R(\theta)^{-1}\dot{\xi}_R = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$$

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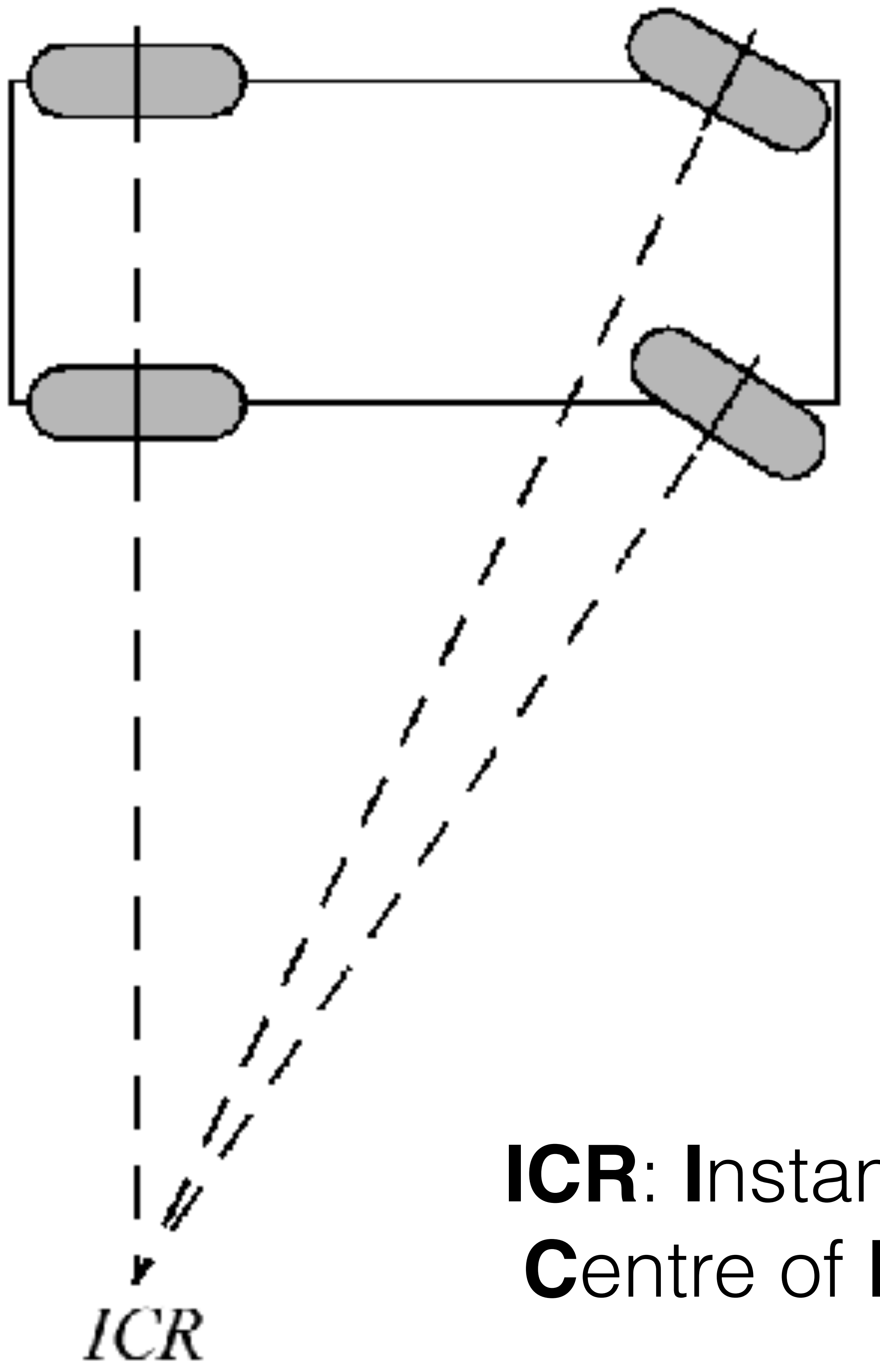
More Complex Scenarios

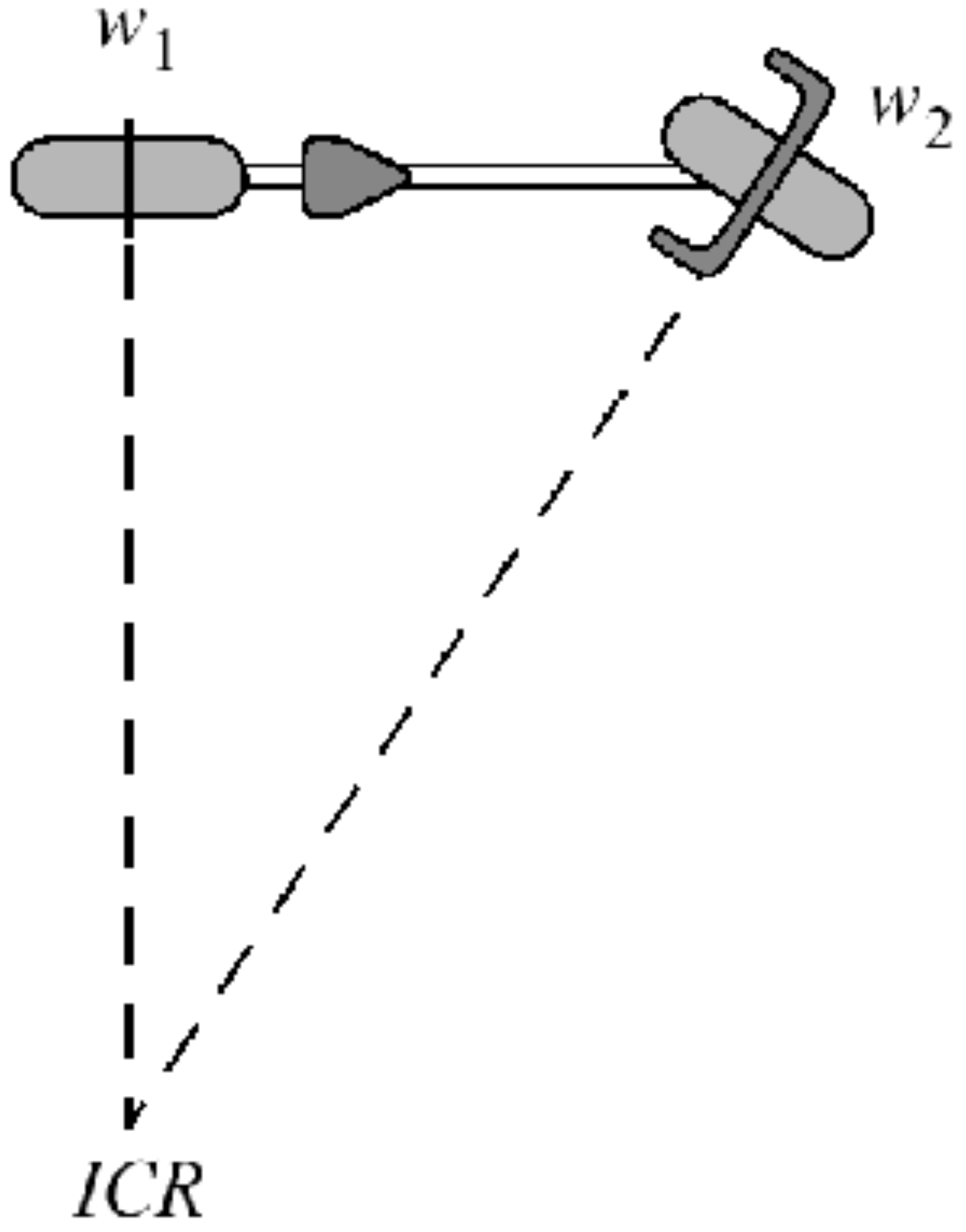
- Making sure the assumptions hold ***imposes constraints on robot***
  - For example, ensuring a rigid chassis.
- Knowing what the assumptions are ***imposes constraints on the applicability of the model***
  - For example, ensuring wheels don't slip.
- Some constraints can be relaxed by using other wheels
  - Eg. castor wheel or Swedish wheel, or using a steering wheel
  - ***But these introduce additional parameters***!

Robot Mobility

- The sliding constraint means that a standard wheel has no lateral motion.
    - Zero motion line through the axis.
- Has to move along a circle whose centre is on the zero motion line



ICR: **I**nstantaneous **C**entre of **R**otation



Robot Mobility

- A differential drive robot has just one line of zero motion.
- Thus its rotation is not constrained
  - It can move in any circle it wants.

- Makes it very easy to move around.
- In general, the manoeuvrability of a robot depends on the number of independent kinematic constraints.
  - Q: How can we formalise this idea?
  - A: ***Degrees of mobility and manoeuvrability***.

Robot Mobility

- Formally we have the notion of a ***degree of mobility***

$$\delta_m = 3 - \text{number of independent kinematic constraints}$$

  - This number is also the number of ***independent*** fixed or steerable standard wheels.
  - ***The independence is important***.

- ***Differential drive*** has two standard wheels, but they are on the same axis.
  - So not both independent.
  - Number of constraints is 1.
- So δ_m = 2 for a differential drive robot
  - Can alter $\dot{x}$ and $\dot{\theta}$ just through wheel velocity.

- ***A bicycle*** has two independent wheels, so two constraints.
  - δ_m = 1
  - Can only alter $\dot{x}$ using wheel velocity.

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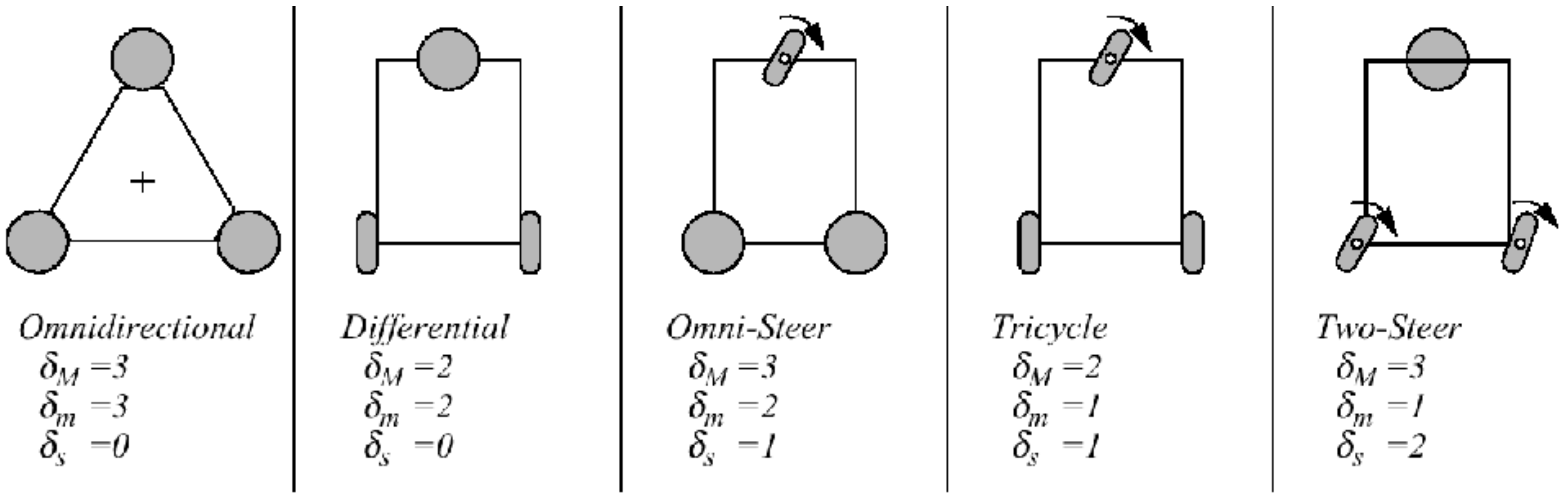
Steerability and manoeuvrability

- Steering has an impact on how the robot moves.
  - The ***degree of steerability*** δ_s is then the number of independent steerable wheels.
    - Note that a steerable standard wheel will both reduce the degree of mobility and increase the degree of steerability.
  - The ***degree of maneuverability*** is $\delta_M = \delta_m + \delta_s$
    - where δ_m tells us how many degrees of freedom a robot can manipulate.
- Two robots with the same δ_M aren't necessarily equivalent (see on).

Robot manoeuvrability

- ***Differential drive*** has no steering wheels.
  - $\delta_s = 0$
  - $\delta_m = 2$
- Thus, $\delta_M = \delta_m + \delta_s = 2$

- ***A bicycle*** has one steering wheel
  - $\delta_s = 1$
  - $\delta_m = 1$
- Thus, $\delta_M = \delta_m + \delta_s = 2$



Common Configurations

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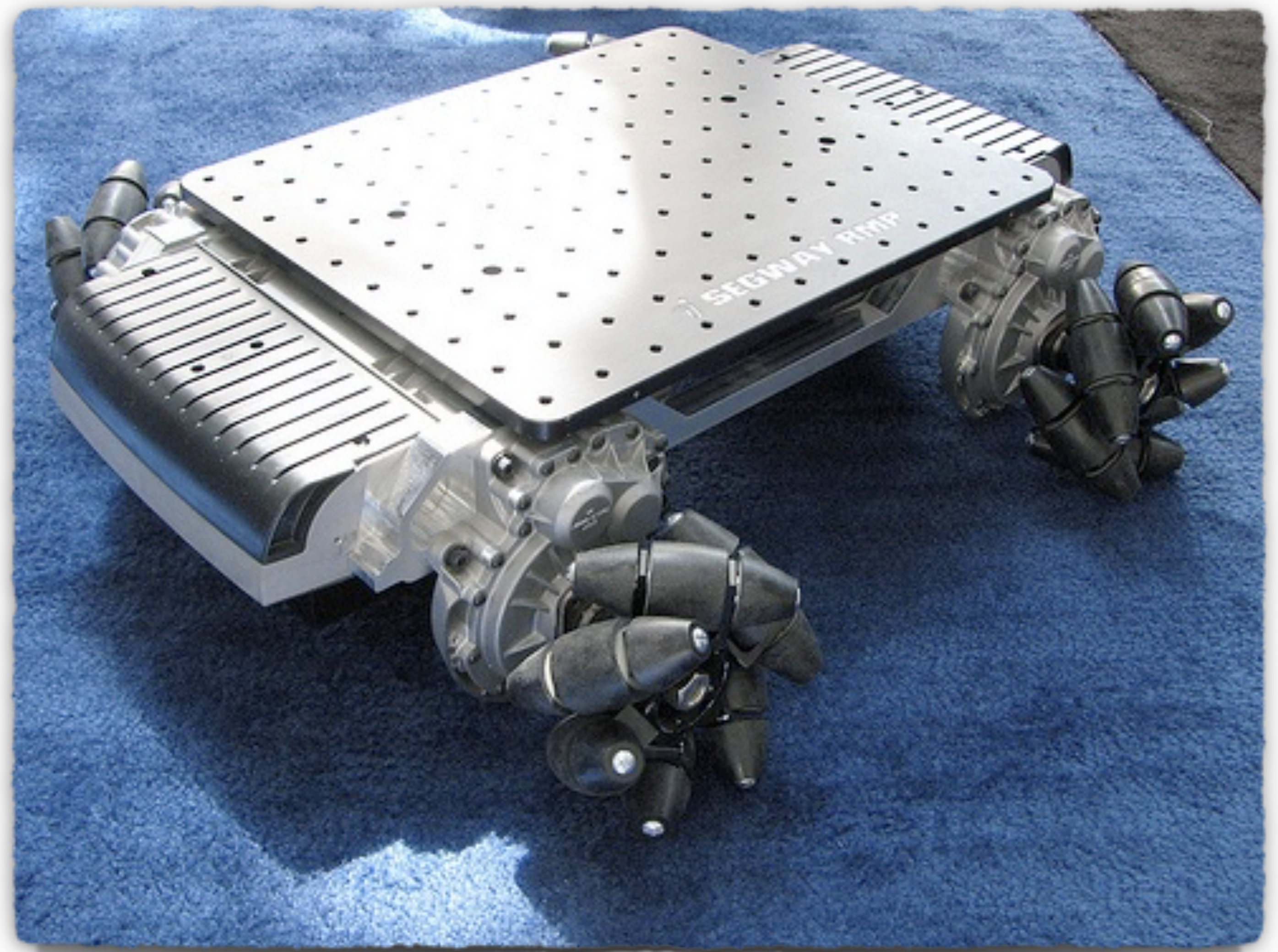
Robot manoeuvrability

- $\delta_M = 2$ is an indication of how easy it is for a robot to move around.
- Compare with the number of DOF in the environment.
  - 3 for the environments we care about.
  - Differential drive and bicycle both have $\delta_M = 2$, but you drive them very differently.
- A bicycle, has a $\delta_M = 2$ yet can position itself anywhere in the plane.
  - But a bicycle only has one DOF that it can control directly (x).
  - **Differential DOF** is always equal to δ_m
- A general inequality:
  - DDOF $\leq \delta_M \leq$ DOF
- A robot with DDOF = DOF is called ***holonomic***



Original Source: M. Wooldridge, S.Parsons, D.Grossi - updated by Terry Payne, Autumn 2016 & 2017

Summary

- This lecture took a brief look at kinematics
  - The business of relating what robots do in the world to what their motors need to be told to do.
  - We did a little maths, but most of the discussion was qualitative.
  - The **Autonomous Mobile Robotics** book goes more into the mathematical detail of establishing kinematic constraints.

- Next time we'll look at more advanced sensors and perception

__Original Source: M. Wooldridge, S.Parsons, D.Grossi - updated by Terry Payne, Autumn 2016 & 2017__