Online Multi-dimensional Dynamic Bin Packing of Unit-Fraction Items

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Abstract. We study the 2-D and 3-D dynamic bin packing problem, in which items arrive and depart at arbitrary times. The 1-D problem was first studied by Coffman, Garey, and Johnson motivated by the dynamic storage problem. Bar-Noy et al. have studied packing of unit fraction items (i.e., items with length $1/k$ for some integer $k \geq 1$), motivated by the window scheduling problem. In this paper, we extend the study of 2-D and 3-D dynamic bin packing problem to unit fractions items. The objective is to pack the items into unit-sized bins such that the maximum number of bins ever used over all time is minimized. We give a scheme that divides the items into classes and show that applying the First-Fit algorithm to each class is 6.7850- and 21.6108-competitive for 2-D and 3-D, respectively, unit fraction items. This is in contrast to the 7.4842 and 22.4842 competitive ratios for 2-D and 3-D, respectively, that would be obtained using only existing results for unit fraction items.

1 Introduction

Bin packing is a classical combinatorial optimization problem that has been studied since the early 70's and different variants continue to attract researchers' attention (see [7, 10, 12]). It is well known that the problem is NP-hard [14]. The problem was first studied in one dimension (1-D), and has been extended to multiple dimensions (d-D, where $d \geq 1$). In d-D packing, the bins have lengths all equal to 1, while items are of lengths in $(0, 1]$ in each dimension. The objective of the problem is to pack the items into a minimum number of unit-sized bins such that the items do not overlap and do not exceed the boundary of the bin. The items are oriented and cannot be rotated.

Extensive work (see [7, 10, 12]) has been done in the offline and online settings. In the offline setting, all the items and their sizes are known in advance. In the online setting, items arrive at unpredictable time and the size is only known when the item arrives. The performance of an online algorithm is measured using competitive analysis [3]. Consider any online algorithm $\mathcal{A}$ with an input I. Let $OPT(I)$ and $\mathcal{A}(I)$ be the maximum number of bins used by the optimal offline algorithm and $\mathcal{A}$, respectively. Algorithm $\mathcal{A}$ is said to be c-competitive if there exists a constant b such that $\mathcal{A}(I) \leq c \cdot OPT(I) + b$ for all I.

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In some real applications, item size is not represented by arbitrary real numbers in $(0, 1]$. Bar-Noy et al. [2] initiated the study of the *unit fraction bin packing problem*, a restricted version where all sizes of items are of the form $\frac{1}{k}$, for some integer k. The problem was motivated by the window scheduling problem [1, 2]. Another related problem is for *power fraction* items, where sizes are of the form $\frac{1}{2^k}$, for some integer k. Bin packing with other restricted form of item sizes includes divisible item sizes [8] (where each possible item size can be divided by the next smaller item size) and discrete item sizes [6] (where possible item sizes are $\{1/k, 2/k, \cdots, j/k\}$ for some $1 \leq j \leq k$). For d-D packing, items of restricted form have been considered, e.g., [16] considered strip packing ([19]) of items with one of the dimensions having discrete sizes and [17] considered bin packing of items where the lengths of each dimension are at most $1/m$, for some integer m. The study of these problems is motivated by applications in job scheduling. As far as we know, unit or power fraction items have not been considered in multi-dimensional packing.

Dynamic Bin Packing. Earlier work concentrated on "static" bin packing, where items do not depart. In potential applications, like warehouse storage, a more realistic setting is the dynamic model, where items arrive and depart dynamically. This natural generalization, known as dynamic bin packing problem, was introduced by Coffman, Garey, and Johnson [9]. The items arrive over time, reside for some period of time, and may depart at arbitrary times. Each item must be packed to a bin from its arrival to its departure. Again, migration to another bin is not allowed, yet rearrangement of items within a bin is allowed. The objective is to minimize the maximum number of bins used over all time. In the offline setting, the sizes, and arrival and departure times of items are known in advance, while in the online setting the sizes and arrival times of items are only known when items arrive, and the departure times are known only when items depart.

Previous Work. The dynamic bin packing problem was first studied in 1-D for general size items by Coffman, Garey and Johnson [9], showing that the First-Fit (FF) algorithm has a competitive ratio lying between 2.75 and 2.897, and a modified First-Fit algorithm is 2.788-competitive. They gave a formula of the competitive ratio of FF when the item size is at most $\frac{1}{k}$. When $k = 2$ and 3, the ratios are 1.7877 and 1.459, respectively. They also gave a lower bound of 2.388 for any deterministic online algorithm, which was improved to 2.5 [5] and then to 2.666 [21]. For unit fraction items, Chan et al. [4] obtained a competitive ratio of 2.4942, which was recently improved by Han et al. to 2.4842 [15], while the lower bound was proven to be 2.428 [4]. Multi-dimensional dynamic bin packing of general size items has been studied by Epstein and Levy [13], who showed that the competitive ratios are 8.5754, 35.346 and $2 \cdot 3.5^d$ for 2-D, 3-D and d-D, respectively. The ratios are then improved to 7.788, 22.788, and 3^d, correspondingly [20]. For 2-D and 3-D general size items, the lower bounds are 3.70301 and 4.85383 [13], respectively. In this case, the lower bounds apply even to unit fraction items.

Table 1. Competitive ratios for general size, unit fraction, and power fraction items. Results obtained in this paper are marked with "[*]".

	1-D	2-D	3-D
General size	2.788 [9]	7.788 [20]	22.788 [20]
Unit fraction	2.4842 [15]	6.7850 [*]	21.6108 [*]
Power fraction	2.4842 [15]	6.2455 [*]	20.0783 [*]

Our Contribution. In this paper, we extend the study of 2-D and 3-D online dynamic bin packing problem to unit and power fraction items. We observe that using the 1-D results on unit fraction items [15], the competitive ratio of 7.788 for 2-D [20] naturally becomes 7.4842, while the competitive ratio of 22.788 for 3-D [20] becomes 22.4842. An immediate question arising is whether we can have an even smaller competitive ratio. We answer the questions affirmatively as follows (see Table 1 for a summary).

- For 2-D, we obtain competitive ratios of 6.7850 and 6.2455 for unit and power fraction items, respectively; and
- For 3-D, we obtain competitive ratios of 21.6108 and 20.0783 for unit and power fraction items, respectively.

We adopt the typical approach of dividing items into classes and analyzing each class individually. We propose several natural classes and define different packing schemes based on the classes[1]. In particular, we show that two schemes lead to better results. We show that one scheme is better than the other for unit fraction items, and vice versa for power fraction items. Our approach gives a systematic way to explore different combinations of classes. One observation we have made is that dividing 2-D items into three classes gives comparable results but dividing into four classes would lead to much higher competitive ratios.

As an attempt to justify the approach of classifying items, we show that, when classification is not used, the performance of the family of any-fit algorithms is unbounded for 2-D general size items. This is in contrast to the case of 1-D packing, where the First-Fit algorithm (without classification) is $O(1)$-competitive [9].

2 Preliminaries

Notations and Definitions. We consider the online dynamic bin packing problem, in which 2-D and 3-D items must be packed into 2-D and 3-D unit-sized bins, respectively, without overflowing. The items arrive over time, reside for some period of time, and may depart at arbitrary times. Each item must be packed into a bin from its arrival to its departure. Migration to another bin is not allowed and the items are oriented and cannot be rotated. Yet, repacking

[1] The proposed classes are not necessarily disjoint while a packing scheme is a collection of disjoint classes that cover all types of items.

Table 2. Types of unit fraction items considered

$\mathrm{T}(1,1)$	$\mathrm{T}(1,\frac{1}{2})$	$\mathrm{T}(\frac{1}{2},1)$	$\mathrm{T}(\frac{1}{2},\frac{1}{2})$	$\mathrm{T}(1,\leq\frac{1}{3})$	$\mathrm{T}(\frac{1}{2},\leq\frac{1}{3})$	$\mathrm{T}(\leq\frac{1}{3},\leq 1)$

Table 3. The 2-D results of [20] for unit-fraction items

Scheme in [20]		
Classes	Types of items	Competitive ratios
Class A	$\mathrm{T}(\leq\frac{1}{3},\leq 1)$	3 [20]
Class B	$\mathrm{T}(\frac{1}{2},1),\mathrm{T}(\frac{1}{2},\frac{1}{2}),\mathrm{T}(\frac{1}{2},\leq\frac{1}{3})$	2 [20]
Class C	$\mathrm{T}(1,1),\mathrm{T}(1,\frac{1}{2}),\mathrm{T}(1,\leq\frac{1}{3})$	2.4842 [15]
Overall	All items	7.4842

of items within the same bin is permitted[2]. The *load* refers to the total area or volume of a set of 2-D or 3-D items, respectively. The objective of the problem is to minimize the total number of bins used over all time. For both 2-D and 3-D, we consider two types of input: unit fraction and power fraction items.

A *general size item* is an item such that the length in each dimension is in $(0,1]$. A *unit fraction (UF) item* is an item with lengths of the form $\frac{1}{k}$, where $k \geq 1$ is an integer. A *power fraction (PF) item* has lengths of the form $\frac{1}{2^k}$, where $k \geq 0$ is an integer.

A packing is said to be *feasible* if all items do not overlap and the packing in each bin does not exceed the boundary of the bin; otherwise, the packing is said to *overflow* and is *infeasible*.

Some of the algorithms discussed in this paper repack existing items (and possibly include a new item) in a bin to check if the new item can be packed into this bin. If the repacking is infeasible, it is understood that we would restore the packing to the original configuration.

For 2-D items, we use the notation $\mathrm{T}(w,h)$ to refer to the type of items with width w and height h. We use '$*$' to mean that the length can take any value at most 1, e.g., $\mathrm{T}(*,*)$ refers to all items. The parameters w (and h) may take an expression $\leq x$ meaning that the width is at most x. For example, $\mathrm{T}(\frac{1}{2},\leq\frac{1}{2})$ refers to the items with width $\frac{1}{2}$ and height at most $\frac{1}{2}$. In the following discussion, we divide the items into seven disjoint types as showed in Table 2.

The bin assignment algorithm that we use for all types of 2-D and 3-D unit and power fraction items is the First-Fit (FF) algorithm. When a new item arrives, if there are occupied bins in which the item can be repacked, FF assigns the new item to the bin which has been occupied for the longest time.

Remark on Existing Result on Unit Fraction Items. Using this notation, the algorithm in [20] effectively classifies unit fraction items into the classes as shown in Table 3. Items in the same class are packed separately, independent of

[2] If rearrangement within a bin is not allowed, one can show that there is no constant competitive deterministic online algorithm.

other classes. The overall competitive ratio is the sum of the competitive ratios of all classes. By the result in [15], the competitive ratio for Class C reduces from 2.788 [9] to 2.4842 [15] and the overall competitive ratio immediately reduces from 7.778 to 7.4842.

Corollary 1. *The* 2*-D packing algorithm in* [20] *is* 7.4842*-competitive for UF items.*

Remarks on Using Classification of Items. To motivate our usage of classification of items, let us first consider algorithms that do not use classification. In the full paper, we show that the family of any-fit algorithms is unbounded for 2-D general size items (Lemma 1). When a new item R arrives, if there are occupied bins in which R can be packed (allowing repacking for existing items), the algorithms assign R to one of these bins as follows: First-Fit (FF) assigns R to the bin which has been occupied for the longest time; Best-Fit (BF) assigns R to the heaviest loaded bin with ties broken arbitrarily; Worst-Fit (WF) assigns R to the lightest loaded bin with ties broken arbitrarily; Any-Fit (AF) assigns R to any of the bins arbitrarily.

Lemma 1. *The competitive ratio of the any-fit family of algorithms (*First-Fit*,* Best-Fit*,* Worst-Fit*, and* Any-Fit*) for the online dynamic bin packing problem of* 2*-D general size items with no classification of items is unbounded.*

When the items are unit fraction and no classification is used, we can show that FF is not c-competitive for any $c < 5.4375$, BF is not c-competitive for any $c < 9$, and WF is not c-competitive for any $c < 5.75$. The results hold even for power fraction items. These results are in contrast to the lower bound of 3.70301 of unit fraction items for any algorithm [13].

Repacking. To determine if an item can be packed into an existing bin, we will need some repacking. Here we make some simple observations about the load of items if repacking is not feasible. We first note the following lemma which is implied by Theorem 1.1 in [18].

Lemma 2 ([18]). *Given a bin with width u and height v, if all items have width at most $\frac{u}{2}$ and height at most v, then any set of these items with total area at most $\frac{uv}{2}$ can fit into the same bin by using Steinberg's algorithm.*

The implication of Lemma 2 is that if packing a new item of width $w \le \frac{u}{2}$ and height h into a bin results in infeasible packing, then the total load of the existing items is at least $\frac{uv}{2} - wh$.

Lemma 3. *Consider packing of two types of items $\mathrm{T}(\frac{1}{2}, \le h)$ and $\mathrm{T}(1, *)$, for some $0 < h < 1$. If we have an item of type $\mathrm{T}(1, h')$ that cannot be packed to an existing bin, then the current load of the bin is at least $1 - \frac{h}{2} - h'$.*

Proof. We first pack all items with width 1, including the new type $\mathrm{T}(1, h')$ item, one by one on top of the previous one. For the remaining space, we divide it into two equal halves each with width $\frac{1}{2}$. We then try to pack the $\mathrm{T}(\frac{1}{2}, \le h)$

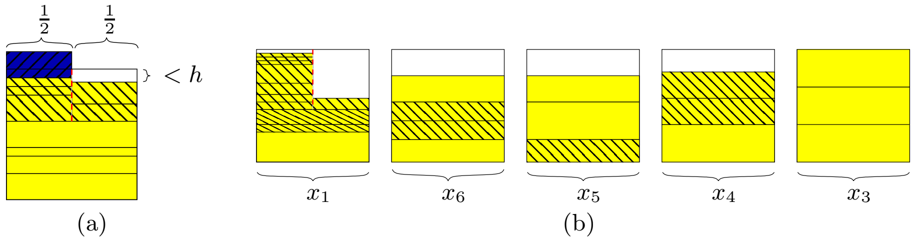

Fig. 1. (a) Infeasible repacking of existing items of types $\mathrm{T}(1, \leq\frac{1}{3})$ and $\mathrm{T}(\frac{1}{2}, \leq\frac{1}{3})$ and a new item of type $\mathrm{T}(1, *)$. The empty space has width $\frac{1}{2}$ and height less than h. (b) Illustration of the proof of Lemma 8. In each set of bins, the shaded items are the item types that do not appear in subsequent bins. For example, items of type $\mathrm{T}(\frac{1}{2}, \leq \frac{1}{3})$ in the first x_1 bins do not appear in the subsequent bins.

Table 4. Values of $\beta\langle x, y\rangle$ for $3 \leq x \leq 6$ and $3 \leq y \leq 6$

$\beta\langle x, y\rangle$	$y = 3$	4	5	6
$x = 3$	1	1	1	1
4	$\frac{3}{4}$	$\frac{5}{6} = \frac{1}{3} + \frac{1}{4} + \frac{1}{4}$	$\frac{5}{6}$	$\frac{11}{12} = \frac{2}{3} + \frac{1}{4}$
5	$\frac{7}{10} = \frac{1}{4} + \frac{1}{4} + \frac{1}{5}$	$\frac{47}{60} = \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$	$\frac{5}{6}$	$\frac{17}{20} = \frac{1}{4} + \frac{3}{5}$
6	$\frac{7}{10}$	$\frac{23}{30} = \frac{1}{6} + \frac{3}{5}$	$\frac{49}{60} = \frac{1}{4} + \frac{1}{6} + \frac{2}{5}$	$\frac{17}{20}$

items into one compartment until it overflows, and then continue packing into the other compartment. The space left in the second compartment has a height less than h, otherwise, the overflow item can be packed there (see Figure 1(a)). As a result, the total load of items is at least $1 - \frac{h}{2}$. Since the new item has a load of h', the total load of existing items is at least $1 - \frac{h}{2} - h'$ as claimed. □

In the case of 1-D packing, Chan et al. [4] have defined the following notion. Let x and y be positive integers. Suppose that a 1-D bin is already packed with some items whose sizes are chosen from the set $\{1, \frac{1}{2}, \ldots, \frac{1}{x}\}$. They defined the notion of the minimum load of such a bin that an additional item of size $\frac{1}{y}$ cannot fit into the bin. We modify this notion such that the set in concern becomes $\{\frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{x}\}$. We define $\beta\langle x, y\rangle$ to be the minimum load of this bin containing items with length at least $\frac{1}{x}$ and at most $\frac{1}{3}$ such that an item of size $\frac{1}{y}$ cannot be packed into this bin. Precisely,

$$\beta\langle x, y\rangle = \min_{3 \leq j \leq x \text{ and } n_j \geq 0} \{\frac{n_3}{3} + \frac{n_4}{4} + \ldots + \frac{n_x}{x} \mid \frac{n_3}{3} + \frac{n_4}{4} + \ldots + \frac{n_x}{x} > 1 - \frac{1}{y}\}.$$

Table 4 shows the values of this function for $3 \leq x \leq 6$ and $3 \leq y \leq 6$.

Table 5. Classifications of 2-D unit fraction items and their competitive ratios

Classes	Types of items	Competitive ratios
Class 1	$\mathrm{T}(\le\frac{1}{3}, \le 1)$	2.8258
Class 2	$\mathrm{T}(1, \le\frac{1}{3})$, $\mathrm{T}(\frac{1}{2}, \le\frac{1}{3})$	1.7804
Class 3	$\mathrm{T}(1,1)$, $\mathrm{T}(1,\frac{1}{2})$, $\mathrm{T}(\frac{1}{2},1)$, $\mathrm{T}(\frac{1}{2},\frac{1}{2})$	2.25
Class 4	$\mathrm{T}(1,\frac{1}{2})$, $\mathrm{T}(1,\le\frac{1}{3})$, $\mathrm{T}(\frac{1}{2},\frac{1}{2})$, $\mathrm{T}(\frac{1}{2},\le\frac{1}{3})$	2.4593
Class 5	$\mathrm{T}(1,1)$, $\mathrm{T}(\frac{1}{2},1)$	1.5

3 Classification of 2-D Unit Fraction Items

Following the idea in [20], we also divide the type of items into classes. In Table 5, we list the different classes we considered in this paper. We propose two packing schemes, each of which makes use of a subset of the classes that are disjoint. The competitive ratio of a packing scheme is the sum of the competitive ratio we can achieve for each of the classes in the scheme. In this section, we focus on individual classes and in the next section, we discuss the two packing schemes. For each class, we use FF (First-Fit) to determine which bin to assign an item. For each bin, we check if the new item can be packed together with the existing items in the bin; this is done by some repacking procedures and the repacking is different for different classes.

Class 5: $\mathbf{T(1,1), T(\frac{1}{2},1)}$

This is a simple case and we skip the details.

Lemma 4. *FF is* 1.5*-competitive for UF items of types* $\mathrm{T}(1,1)$ *and* $\mathrm{T}(\frac{1}{2},1)$.

Class 3: $\mathbf{T(1,1), T(1,\frac{1}{2}), T(\frac{1}{2},1), T(\frac{1}{2},\frac{1}{2})}$

We now consider Class 3 for which both the width and height are at least $\frac{1}{2}$.

Lemma 5. *FF is* 2.25*-competitive for UF items of types* $\mathrm{T}(1,1), \mathrm{T}(1,\frac{1}{2}), \mathrm{T}(\frac{1}{2},1),$ $\mathrm{T}(\frac{1}{2},\frac{1}{2})$.

Proof. Suppose the maximum load at any time is n. Then OPT uses at least n bins. Let x_1 be the last bin that FF ever packs a $\mathrm{T}(\frac{1}{2},\frac{1}{2})$-item, $x_1 + x_2$ for $\mathrm{T}(1,\frac{1}{2})$ and $\mathrm{T}(\frac{1}{2},1)$, and $x_1 + x_2 + x_3$ for $\mathrm{T}(1,1)$. When FF packs a $\mathrm{T}(\frac{1}{2},\frac{1}{2})$-item to bin-$x_1$, all the $x_1 - 1$ before that must have a load of 1. Therefore, $(x_1 - 1) + \frac{1}{4} \le n$. When FF packs a $\mathrm{T}(1,\frac{1}{2})$ or $\mathrm{T}(\frac{1}{2},1)$-item to bin-$(x_1 + x_2)$, all the bins before that must have a load of $\frac{1}{2}$. Hence, $\frac{x_1+x_2}{2} \le n$. When FF packs a $\mathrm{T}(1,1)$-item to bin-$(x_1 + x_2 + x_3)$, the first x_1 bins must have a load of at least $\frac{1}{4}$, the next x_2 bins must have a load of at least $\frac{1}{2}$, and the last $x_3 - 1$ bins must have a load of 1. Therefore, $\frac{x_1}{4} + \frac{x_2}{2} + (x_3 - 1) + 1 \le n$. The maximum value of $x_1 + x_2 + x_3$ is obtained by setting $x_1 = x_2 = n$ and $x_3 = \frac{n}{4}$. Then, $x_1 + x_2 + x_3 = 2.25n \le 2.25OPT$. □

Class 2: $\mathbf{T(1, \leq\frac{1}{3}), T(\frac{1}{2}, \leq\frac{1}{3})}$

We now consider items whose width is at least $\frac{1}{2}$ and height is at most $\frac{1}{3}$. For this class, the repack when a new item arrives is done according to the description in the proof of Lemma 3. We are going to show that FF is 1.7804-competitive for Class 2.

Suppose the maximum load at any time is n. Let x_1 be the last bin that FF ever packs a $\mathrm{T}(\frac{1}{2}, \leq\frac{1}{3})$-item. Using the analysis in [9] for 1-D items with size at most $\frac{1}{3}$, one can show that $x_1 \leq 1.4590n$.

Lemma 6 ([9]). *Suppose we are packing UF items of types* $\mathrm{T}(1, \leq\frac{1}{3}), \mathrm{T}(\frac{1}{2}, \leq\frac{1}{3})$ *and the maximum load over time is n. We have $x_1 \leq 1.4590n$, where x_1 is the last bin that FF ever packs a* $\mathrm{T}(\frac{1}{2}, \leq\frac{1}{3})$*-item.*

Lemma 6 implies that FF only packs items of $\mathrm{T}(1, \leq\frac{1}{3})$ in bin-y for $y > 1.459n$. The following lemma further asserts that the height of these items is at least $\frac{1}{6}$.

Lemma 7. *Suppose we are packing UF items of types* $\mathrm{T}(1, \leq\frac{1}{3}), \mathrm{T}(\frac{1}{2}, \leq\frac{1}{3})$ *and the maximum load over time is n. Any item that is packed by FF to bin-y, for $y > 1.459n$, must be of type $\mathrm{T}(1, h)$, where $\frac{1}{6} \leq h \leq \frac{1}{3}$.*

Proof. Suppose on the contrary that FF packs a $\mathrm{T}(1, \leq \frac{1}{7})$-item in bin-$y$ for $y > 1.459n$. This means that packing the item in any of the first $1.459n$ bins results in an infeasible packing. By Lemma 3, with $h = \frac{1}{3}$ and $h' = \frac{1}{7}$, the load of each of the first $1.459n$ bins is at least $1 - \frac{1}{6} - \frac{1}{7} = 0.69$. Then the total is at least $1.459n \times 0.69 > 1.0067n$, contradicting that the maximum load at any time is n. Therefore, the lemma follows. □

Lemma 8. *FF is* 1.7804*-competitive for UF items of types* $\mathrm{T}(1, \leq\frac{1}{3})$, $\mathrm{T}(\frac{1}{2}, \leq\frac{1}{3})$.

Proof. Figure 1(b) gives an illustration. Let $(x_1 + x_6), (x_1 + x_6 + x_5), (x_1 + x_6 + x_5 + x_4)$, and $(x_1 + x_6 + x_5 + x_4 + x_3)$ be the last bin that FF ever packs a $\mathrm{T}(1, \frac{1}{6})$-, $\mathrm{T}(1, \frac{1}{5})$-, $\mathrm{T}(1, \frac{1}{4})$-, and $\mathrm{T}(1, \frac{1}{3})$- item, respectively. When FF packs a $\mathrm{T}(1, \frac{1}{6})$-item to bin-$(x_1 + x_6)$, the load of the first x_1 is at least $1 - \frac{1}{6} - \frac{1}{6} = \frac{2}{3}$, by Lemma 3. By Lemma 7, only type $\mathrm{T}(1, k)$-item, for $\frac{1}{6} \leq k \leq \frac{1}{3}$, could be packed in the x_6 bins. These items all have width 1 and thus can be considered as 1-D case. Therefore, when we cannot pack a $\mathrm{T}(1, \frac{1}{6})$-item, the current load must be at least $\beta\,\langle 6, 6\rangle$. Then we have $x_1(\frac{2}{3}) + x_6\,\beta\,\langle 6, 6\rangle \leq n$. Similarly, we have

1. $x_1(\frac{2}{3}) + x_6\,\beta\,\langle 6, 6\rangle \leq n$,
2. $x_1(1 - \frac{1}{6} - \frac{1}{5}) + x_6\,\beta\,\langle 6, 5\rangle + x_5\,\beta\,\langle 5, 5\rangle \leq n$,
3. $x_1(1 - \frac{1}{6} - \frac{1}{4}) + x_6\,\beta\,\langle 6, 4\rangle + x_5\,\beta\,\langle 5, 4\rangle + x_4\,\beta\,\langle 4, 4\rangle \leq n$,
4. $x_1(1 - \frac{1}{6} - \frac{1}{3}) + x_6\,\beta\,\langle 6, 3\rangle + x_5\,\beta\,\langle 5, 3\rangle + x_4\,\beta\,\langle 4, 3\rangle + x_3\,\beta\,\langle 3, 3\rangle \leq n$.

We note that for each inequality, the coefficients are increasing, e.g., for (1), we have $\frac{2}{3} \leq \beta\,\langle 6, 6\rangle = \frac{17}{20}$, by Table 4. Therefore, the maximum value of $x_1 + x_6 + x_5 + x_4 + x_3$ is obtained by setting the maximum possible value of x_6

Table 6. Competitive ratios for 2-D unit fraction items

2DDynamicPackUFS1		
Classes	Types of items	Competitive ratios
Class 1	$\mathrm{T}(\leq\frac{1}{3}, \leq 1)$	2.8258
Class 4	$\mathrm{T}(1, \frac{1}{2})$, $\mathrm{T}(1, \leq\frac{1}{3})$, $\mathrm{T}(\frac{1}{2}, \frac{1}{2})$, $\mathrm{T}(\frac{1}{2}, \leq\frac{1}{3})$	2.4593
Class 5	$\mathrm{T}(1, 1)$, $\mathrm{T}(\frac{1}{2}, 1)$	1.5
Overall	All of the above	6.7850

satisfying (1), and then the maximum possible value of x_5 satisfying (2), and so on. Using Table 4, we compute the corresponding values as $1.4590n$, $0.0322n$, $0.0597n$, $0.0931n$ and $0.1365n$, respectively. As a result, $x_1 + x_6 + x_5 + x_4 + x_3 \leq 1.7804n \leq 1.7804OPT$. □

Class 1: $\mathbf{T}(\leq\frac{1}{3}, \leq 1)$

Items of type $\mathrm{T}(\leq\frac{1}{3}, \leq 1)$ are further divided into three subtypes: $\mathrm{T}(\leq\frac{1}{3}, \leq\frac{1}{3})$, $\mathrm{T}(\leq\frac{1}{3}, \frac{1}{2})$, and $\mathrm{T}(\leq\frac{1}{3}, 1)$. We describe how to repack these items and leave the analysis in the full paper.

1. When the new item is $\mathrm{T}(\leq\frac{1}{3}, \leq\frac{1}{3})$, we use Steinberg's algorithm [18] to repack the new and existing items. Note that the item width satisfies the criteria of Lemma 2.
2. When the new item is $\mathrm{T}(\leq\frac{1}{3}, \frac{1}{2})$ or $\mathrm{T}(\leq\frac{1}{3}, 1)$ and the bin contains $\mathrm{T}(\leq\frac{1}{3}, \leq\frac{1}{3})$-item, we divide the bin into two compartments, one with width $\frac{1}{3}$ and the other $\frac{2}{3}$ and both with height 1. We reserve the small compartment for the new item and try to repack the existing items in the large compartment using Steinberg's algorithm. This idea originates from [20].
3. When the new item is $\mathrm{T}(\leq\frac{1}{3}, \frac{1}{2})$ or $\mathrm{T}(\leq\frac{1}{3}, 1)$ and the bin does not contain $\mathrm{T}(\leq\frac{1}{3}, \leq\frac{1}{3})$-item, we use the repacking method as in Lemma 3 but with the width becoming the height and vice versa. Note that this implies that Lemma 8 applies for these items.

Lemma 9. *FF is* 2.8258*-competitive for UF items of type* $\mathrm{T}(\leq\frac{1}{3}, \leq 1)$.

Class 4: $\mathbf{T}(1, \frac{1}{2}), \mathbf{T}(1, \leq\frac{1}{3}), \mathbf{T}(\frac{1}{2}, \frac{1}{2}), \mathbf{T}(\frac{1}{2}, \leq\frac{1}{3})$

The analysis of Class 4 follows a similar framework as in Class 2. We state the result (Lemma 10) and leave the proof in the full paper.

Lemma 10. *FF is* 2.4593*-competitive for UF items of types* $\mathrm{T}(1, \frac{1}{2})$, $\mathrm{T}(1, \leq\frac{1}{3})$, $\mathrm{T}(\frac{1}{2}, \frac{1}{2})$, $\mathrm{T}(\frac{1}{2}, \leq\frac{1}{3})$.

4 Packing of 2-D Unit Fraction Items

Our algorithm, named as 2DDynamicPackUF, classifies items into classes and then pack items in each class independent of other classes. In each class, FF is

Table 7. Competitive ratios for 2-D power fraction items. Marked with [*] are the competitive ratios that are reduced as compared to unit fraction items.

2DDynamicPackPF		
Class	Types of items	Competitive ratios
Class 1	T($\le\frac{1}{4}, \le 1$)	2.4995 [*]
Class 2	T($1, \le\frac{1}{4}$), T($\frac{1}{2}, \le\frac{1}{4}$)	1.496025 [*]
Class 3	T($1, 1$), T($1, \frac{1}{2}$), T($\frac{1}{2}, 1$), T($\frac{1}{2}, \frac{1}{2}$)	2.25
Overall	All items	6.2455

used to pack the items as described in Section 3. In this section, we present two schemes and show their competitive ratios.

Table 6 shows the classification and associated competitive ratios for 2DDynamicPackUFS1. This scheme contains Classes 1, 4, and 5, covering all items.

Theorem 1. 2DDynamicPackUFS1 *is* 6.7850-*competitive for* 2-*D UF items.*

Scheme 2DDynamicPackUFS2 has a higher competitive ratio than Scheme 2DDynamicPackUFS1, nevertheless, Scheme 2DDynamicPackUFS2 has a smaller competitive ratio for power fraction items to be discussed in the next section. 2DDynamicPackUFS2 contains Classes 1, 2, and 3, covering all items.

Lemma 11. 2DDynamicPackUFS2 *is* 6.8561-*competitive for* 2-*D UF items.*

5 Adaptations to Other Scenarios

In this section we extend our results to other scenarios.

2-D Power Fraction Items. Table 7 shows a scheme based on 2DDynamicPackUFS2 for unit fraction items and the competitive ratio is reduced to 6.2455.

Theorem 2. 2DDynamicPackPF *is* 6.2455-*competitive for* 2-*D PF items.*

3-D Unit and Power Fraction Items. The algorithm in [20] effectively classifies the unit fraction items as shown in Table 8(a). The overall competitive ratio reduces from 22.788 to 21.6108. For power fraction items we slightly modify the classification for 3-D items, such that boundary values of $\frac{1}{3}$ are replaced by $\frac{1}{4}$. Table 8(b) details this classification. The overall competitive ratio reduces to 20.0783. We state the following theorem and leave the proof in the full paper.

Theorem 3. (1) *Algorithm* 3DDynamicPackUF *is* 21.6108-*competitive for UF items and* (2) *algorithm* 3DDynamicPackPF *is* 20.0783-*competitive for PF items.*

Table 8. (a) Competitive ratios for 3-D UF items. [*] This result uses Theorem 1. [**] This result uses Lemma 9. (b) Competitive ratios for 3-D PF items. [*] This result uses Theorem 2. [**] This result uses the competitive ratio of Class 1 2-D PF items.

(a)

3DDynamicPackUF [20]		
Classes	Types of items	Competitive ratios
Class 1	T($> \frac{1}{2}, *, *$)	6.7850 [*]
Class 2	T($\leq \frac{1}{2}, > \frac{1}{2}, *$)	4.8258 [**]
Class 3	T($\leq \frac{1}{2}, (\frac{1}{3}, \frac{1}{2}], *$)	4
Class 4	T($\leq \frac{1}{2}, \leq \frac{1}{3}, *$)	6
Overall	All items	21.6108

(b)

3DDynamicPackPF		
Classes	Types of items	Competitive ratios
Class 1	T($> \frac{1}{2}, *, *$)	6.2455 [*]
Class 2	T($\leq \frac{1}{2}, > \frac{1}{2}, *$)	4.4995 [**]
Class 3	T($\leq \frac{1}{2}, (\frac{1}{4}, \frac{1}{2}], *$)	4
Class 4	T($\leq \frac{1}{2}, \leq \frac{1}{4}, *$)	5.334
Overall	All items	20.0783

6 Conclusion

We have extended the study of 2-D and 3-D dynamic bin packing problem to unit and power fraction items. We have improved the competitive ratios that would be obtained using only existing results for unit fraction items from 7.4842 to 6.7850 for 2-D, and from 22.4842 to 21.6108 for 3-D. For power fraction items, the competitive ratios are further reduced to 6.2455 and 20.0783 for 2-D and 3-D, respectively. Our approach is to divide items into classes and analyzing each class individually. We have proposed several classes and defined different packing schemes based on the classes. This approach gives a systematic way to explore different combinations of classes.

An open problem is to further improve the competitive ratios for various types of items. The gap between the upper and lower bounds could also be reduced by improving the lower bounds. Another problem is to consider multi-dimensional bin packing. For d-dimensional static and dynamic bin packing, for $d \geq 2$, the competitive ratio grows exponentially with d. Yet there is no matching lower bound that also grows exponentially with d. It is believed that this is the case [11] and any such lower bound would be of great interest.

Another direction is to consider the packing of unit fraction and power fraction squares, where all sides of an item are the same length. We note that the competitive ratio for the packing of 2-D unit fraction square items would reduce to 3.9654 compared to the competitive ratio of 2-D general size square items of 4.2154 [13]. For 3-D unit fraction squares, this would reduce to 5.24537 compared to 5.37037 for 3-D general size squares [13].

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