LOWER BOUNDS ON THE COMPLEXITY OF 1-TIME ONLY BRANCHING PROGRAMS

(PRELIMINARY VERSION)

Paul. E. Dunne
Department Of Computer Science
University Of Warwick
Coventry, CV4 7AL
Great Britain

1 Introduction

A *branching program* (*BP*) is a directed acyclic graph where each node has out-degree 2 or zero. Nodes with out-degree equal to zero are called *sinks* and are labelled with boolean constants. The remaining nodes are labelled with boolean variables taken from a set $X=\{x_1,\ldots,x_n\}$. There is a distinguished node, called the *root* which has in-degree equal to zero. A *BP* computes an n-argument boolean function f as follows: Starting at the root, the value of the variable labelling the current node is tested, if it is zero (one) the next node tested is the left (resp. right) descendant of the current node. The *BP* computes f if and only if $\forall \alpha \in \{0,1\}^n$ the path traced from the root under α halts at a sink labelled $f(\alpha)$. The natural complexity measure for a branching program is the number of non-sink nodes. Branching programs are one example of the many forms of restricted boolean networks introduced in attempts to account for the complexity of realising specific boolean functions. Cobham [2] has demonstrated that branching program *depth* and *capacity* (i.e. $log_2(BP\text{-}size)$) are lower bounds on Time and Space in any reasonable model of sequential computation.

In order to acquire insight about arbitrary branching programs, a number of restricted models have been considered (e.g Borodin *et alia* [1], Masek [3]). Among these is the "1-time only" model (BP_1) studied by Masek [3], Pudlak [4], and Wegener [5], [6]. This imposes the constraint that on any path from the root to a sink each variable is tested at most once. Thus a BP_1 has depth at most n. In [6] Wegener proved exponential lower bounds on the BP_1-complexity of certain clique functions. In Section(2) of this paper we show how the argument used may be generalised to yield lower bounds on arbitrary boolean functions. In the remainder of the paper we apply the results of Section(2) to obtain exponential lower bounds on the BP_1-complexity of the directed and undirected Hamiltonian circuit functions and perfect matchings. The lower bounds obtained for the perfect matching and undirected hamiltonian circuit predicates are, to date, to largest established for explicitly defined functions in the model.

<u>Definitions</u>

Let **T** be a BP_1 computing some boolean function f over **X**. Let v and w be internal nodes of **T** (i.e. non-sink nodes). v and w may be *merged* if and only if all the input

branches of v may t
computation of f.

A variable x_i is
tested at some node
includes the right (le
path if no node on th

Below, unless oth

$X_n = \{x_{ij} \mid 1 \leq i <$
n-vertex graph in whi

K_n denotes the c
$(\varepsilon(G,n))$ is the minim
graph which does not

A graph is *k-regu*

<u>Notation</u>

$BP_1(f)$ = *1-time o*

$G(V,E)$: Arbitrary

$H(A,B,E)$: Arbitrary

$\delta(G)$: Degree of gr

If $C \subset V(G)$ then:

$\underline{N}(C)$ = $\{v$ *in* $V(G)$-

2 General Lower

Definition 1: Let
all its arguments. $\theta($
$\forall\, x \in X,\ \forall\, Y \subseteq X-\{x\},$

$\exists$ *a partial assign*

Since it will be tr
we shall assume that
demonstrates how $\theta(f)$
affected by this assum

branches of v may become input branches of w (or *vice-versa*) without affecting the computation of f.

A variable x_i is *positive* (*negative*) on a path from the root to v, if x_i has been tested at some node on this path and found to have the value 1 (0), i.e. the path includes the right (left) branch of the x_i testing node. A variable x_i is *untested* on a path if no node on the path is labelled x_i.

Below, unless otherwise stated, *graph* will mean "undirected simple graph".

$X_n = \{x_{ij} \mid 1 \leq i < j \leq n\}$ will denote a set of $n(n-1)/2$ boolean variables. $G(X_n)$ is the n-vertex graph in which there is an edge (i,j) if and only if $x_{ij}=1$.

K_n denotes the complete graph on n vertices. The *extremal number* of a graph G ($\varepsilon(G,n)$) is the minimum number of edges which must be removed from K_n to leave a graph which does not contain G as a subgraph.

A graph is *k-regular* if every vertex has degree k.

Notation

$BP_1(f)$ = *1-time only branching program complexity of f*

$G(V,E)$: Arbitrary n-vertex undirected graph with vertex set $V(G)$ and edge set $E(G)$.

$H(A,B,E)$: Arbitrary bipartite graph with vertex sets A and B.

$\delta(G)$: Degree of graph G

If $C \subset V(G)$ then:

$\underline{N}(C)$ = $\{v \text{ in } V(G)-C \mid v \text{ is adjacent to some vertex of } C\}$

2 General Lower Bounds

Definition 1: Let f be a (non-constant) boolean function over X, which depends on all its arguments. $\theta(f)$ is defined to the greatest value α (where $0 \leq \alpha \leq n-1$) such that: $\forall\ x \in X,\ \forall\ Y \subseteq X-\{x\}$, with $|Y| \leqslant \alpha$

$\exists$ *a partial assignment*, π, to the variables of $X - Y - \{x\}$, for which:

$$f^{\pi}(x, Y) = x \text{ or } f^{\pi}(x, Y) = \neg x$$

Since it will be true for the functions considered in Sect(3), for the sake of brevity we shall assume that the former case always holds for each $x \in X$. The following lemma demonstrates how $\theta(f)$ relates to BP_1's computing f, and clearly its correctness is not affected by this assumption.

Lemma 1: Let T be any BP_1 computing $f(X)$. If v and w are distinct nodes of T such that on some path p_v to v and on some path p_w to w at most $\theta(f)$ variables have been tested then neither v nor w is a sink of T and v and w cannot be merged.

Proof: Note: A special case of this result is at the core of Wegener's lower bound for clique functions. The proof below is essentially the same argument originally employed by Wegener in [6].

Let γ,δ be the partial assignments used to traverse the paths from the root of T to v,w respectively. Let V be the set of variables tested on the path p_v, excluding the variable labelling v. Similarly let W be the set of variables tested on the path p_w, excluding the variable labelling w. So $|V|, |W| \leq \theta(f)$. Clearly neither v nor w can be a sink, since f can still attain the value 0 or 1. Furthermore if v, w lie on the same path from the root then they cannot be merged without creating a cycle in T. It follows that there is some $x_i \in V \cap W$ which has been postively tested at v and negatively tested at w. Now assume that v, w *can* be merged. Then we claim that $V=W$. For suppose there is some $x_j \in V$ such that $x_j \notin W$. Then since: $f(X) = f(x_j, W, X-W-\{x_j\})$, from the definition of $\theta(f)$ there exists π_δ such that $f^{|\delta,\pi_\delta} = x_j$. Therefore x_j would have to be retested, which is forbidden since T is a 1-time only branching program. Thus $V \subseteq W$ and by a similar argument $W \subseteq V$. So as claimed if v,w can be merged then $V=W$

Now consider the variable x_i. Then: $f(X) = f(x_i, V-\{x_i\}, X-V)$ and since $|V-\{x_i\}| \leq \theta(f)-1$, there exists a partial assignment π to $X-V$ such that:

$f^{|\delta,\pi} = 1$ since $x_i=1$ under δ ; $f^{|\gamma,\pi} = 0$ since $x_i=0$ under γ

This contradiction establishes that v,w cannot be merged. □

The main result of this section is:

Theorem 2: If f is a boolean function over X then: $BP_1(f) \geq 2^{\theta(f)} - 1$

Proof: From the lemma the first $\theta(f)$ levels of any T computing f, must be a complete binary tree. Thus: $BP_1(f) \geq 2^{\theta(f)} - 1$ □

3 The BP_1 Complexity of Some Subgraph Isomorphism Problems

Definition 2: Let $H(A,B,E)$ be a k-regular bipartite graph. H is *neighbourhood restricted* (NR) if and only if:

$\forall\ 1 \leq p \leq n/2-k$:

$\exists \underline{\alpha}_p = \{\alpha_1,\ldots,\alpha_p\} \subseteq A$ such that:

$|\underline{N}(\underline{\alpha}_p)| \leq p + c_0$

$(\alpha_{p+1},\beta_{p+1}) \in E(H)$ for some $\alpha_{p+1} \in A-\underline{\alpha}_p$, $\beta_{p+1} \in B-\underline{N}(\underline{\alpha}_p)$

The subgraph $H^*(V_{rem},E^*)$ of $H(A,B,E)$ where V_{rem}, E^* are given by:

$V(H)-\{\underline{\alpha}_p, \alpha_{p+1}, \beta_{p+1}\} - \underline{N}(\underline{\alpha}_p, \alpha_{p+1}) - \underline{N}(\beta_{p+1})$

$(V_{rem} \times V_{rem}) \cap E(H)$

has $\varepsilon(H^*, |V_{rem}|) > |V_{rem}|-c_1$. Where c_0 and c_1 are constants.

Definition 3: Let $\underline{R} = \{\underline{R}_{i_1}, \underline{R}_{i_2}, \ldots, \underline{R}_{i_j}, \ldots\}$ (where $i_1<i_2<\ldots<i_j<\ldots$) be an infinite family of NR-graphs. (With $|V(\underline{R}_{i_j})|=i_j$).

An n-vertex graph, $G(V,E)$ is an $\underline{R}$-graph if and only if $\underline{R}_n \in \underline{R}$ is a subgraph of G.

For any family $\underline{R}$ as above, $RG\{0,1\}^{n(n-1)/2} \rightarrow \{0,1\}$ is the (monotone) boolean function, defined for $n \in \{i_1,\ldots,i_j,\ldots\}$ to be 1 if and only if $G(X_n)$ is an $\underline{R}$-graph.

As we shall prove below, the undirected hamiltonian circuit function, for graphs with an even number of vertices, and the Perfect Matching function are both expressible as $\underline{R}$-graph predicates for two particular families of NR-graphs.

Theorem 3: For any family of NR-graphs, with $\delta(\underline{R}_n)=k$:

$\theta(RG(X_n)) \geq n/2-k-c$

where $c=max\{c_0, c_1\}$, the constants of Definition(3).

Before giving a proof of this theorem, we illustrate how it is applied to the functions above.

Corollary 4: $UHC(X_n)$ is the monotone boolean function which is 1 if and only if the graph $G(X_n)$ contains an undirected hamiltonian circuit.

$$BP_1(UHC(X_n)) \geq 2^{n/2-6}$$

Proof: Only the case where n is even need be considered. Let $\mathfrak{HC}$ be the family of graphs:

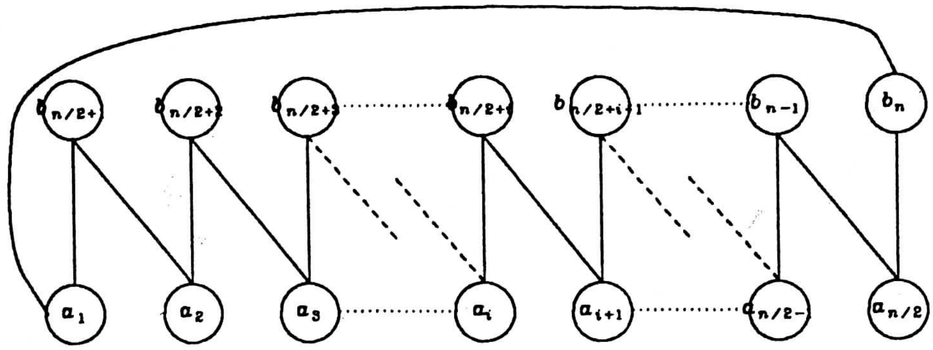
$$\{C_4, C_6, \ldots, C_{2i}, \ldots\}$$

in which the i'th member is the $2i$ vertex *cycle graph*. Each C_n is 2-regular and bipartite (since n is even). (Fig(1a)) C_n is also neighbourhood restricted, since (assuming the labelling of Fig(1a)):

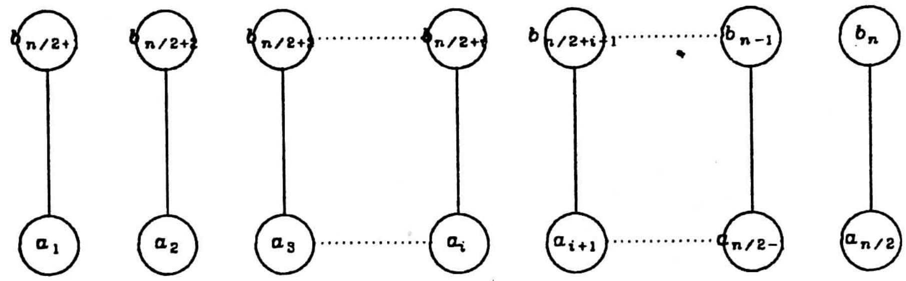
$\forall\ 1 \leq p \leq n/2-1$
$|\underline{N}\ (a_1, a_2, \ldots a_p)| \leq p+1$

The subgraph of C_n consisting of the vertices:

$$\{a_{p+1}, a_{p+2}, \ldots, a_{n/2}, \ldots, b_{n-1}\}$$



1(a) $\mathcal{C}_n$



1(b) $\mathcal{PM}_n$

Figure 1

is just a hamilt

This is pr
this lower bour
set of n-6 edg
vertex v such t

[(

Then F contain
$\delta(v)<n-1$, the s
hamiltonian pa
in which this p
connected by a

It follows
removed from
consisting of t
certainly a ham
found. Thus F

Theorem(2)

It may be obse
circuit function.

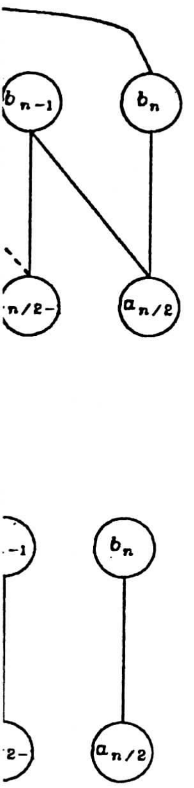
Corollary 5
matching if an
every vertex is
monotone boole
matching.

Proof: Let

Clearly, PM
follows if:

To prove th
is: $(n-1).(n-3)...$
most: $(n-3).(n-$
$BP_1(PM(X_n))$ follo

Proof: (of T



is just a hamiltonian path. The corollary therefore follows if:

$$\varepsilon(n\text{-vertex hamiltonian path}, n) > n-6$$

This is proved by induction on $n \geq 2$. The base is immediate so assume this lower bound on ε holds for all values less than n. Consider K_n with any set of n-6 edges removed to leave a new n-vertex graph F. If F contains a vertex v such that:

$$\lceil(n-3)/2\rceil < \delta(v) < n-1 \quad (\lceil..\rceil \text{ denotes } \textit{ceiling function})$$

Then F contains a hamiltonian path. For, by the inductive hypothesis since $\delta(v)<n-1$, the subgraph consisting of $V(F)-v$ and their incident edges contains a hamiltonian path, and as $\delta(v)> \lceil(n-3)/2\rceil$, so v must be adjacent to the vertex in which this path starts or finishes, or v is adjacent to two vertices which are connected by an edge in this path.

It follows that the only case to be considered is when all the edges removed from K_n are incident to a single vertex v. But now the subgraph consisting of the remaining vertices and their incident edges is K_{n-1}, and certainly a hamiltonian path starting in a vertex to which v is adjacent can be found. Thus F contains a hamiltonian path and the lower bound on ε follows.

Theorem(2) and Theorem(3) yield the lower bound on $BP_1(UHC(X_n))$. □

It may be observed that a similar lower bound holds for the *directed* hamiltonian circuit function.

Corollary 5: An n-vertex graph G(V,E), where n is even, contains a *perfect matching* if and only if there exists a subset of E(G) of size $n/2$ such that every vertex is an endpoint of exactly one edge in this subset. $PM(X_n)$ is the monotone boolean function which takes the value 1 if G(X_n) contains a perfect matching.

$$BP_1(PM(X_n)) \geq 2^{n/2-2}$$

Proof: Let $\mathfrak{PM}$ be the family of graphs (Fig(1b)):

$$\{PM_2, PM_4, \ldots, PM_{2i}, \ldots\}$$

Clearly, PM_n is neighbourhood restricted, with $\mathbf{c}_0$=0. Thus the Corollary follows if:

$$\varepsilon(PM_n, n) > n-2$$

To prove this, observe that the number of distinct perfect matchings in K_n is: (n-1).(n-3).....(3).(1). However removing any one edge from K_n can destroy at most: (n-3).(n-5)....(3).(1) perfect matchings. The lower bound on ε and hence $BP_1(PM(X_n))$ follows. □

Proof: (of Theorem 3)

Let $Y \subset X_n$ of size at most $n/2-k-c$. Without loss of generality, suppose the edge $(1,2) \notin Y$. We shall prove that the edges in $x_n - Y - (1,2)$ may be fixed in such a way that the resulting graph is an $\underline{R}$-graph if and only if it contains the edge (1,2). Let:

$\Gamma_i = \{v \in V(G) | \ (i,v) \in Y\}$ where $i=1$ or 2

$NULL = \{v \in V(G) - \{1,2\} | \ (j,v) \notin Y \ \forall \ 1 \leq j \leq n\}$

NULL is the set of vertices of G none of whose incident edges occur in Y. Now:

$$|NULL| \geq n-2-|\Gamma_1 \cup \Gamma_2|-2(n/2-k-c-|\Gamma_1|-|\Gamma_2|)$$
$$\geq |\Gamma_1| + |\Gamma_2| + 2(k-1)+2c$$

Consider the following mapping from $V(G)$ to $A(R_n) \cup B(R_n)$ $(=V(R_n))$.

M1) The vertices in $\Gamma_1 \cup \Gamma_2$ are mapped to $\underline{\alpha}_p$ in $V(R_n)$. Here $p = |\Gamma_1 \cup \Gamma_2| \leq n/2-k-c$, and $\underline{\alpha}_p$ is the "neighbourhood restricted" set of Definition(2).

M2) A subset $Match_Y$ of *NULL*, with size $|\underline{N}(\underline{\alpha}_p)|$ is mapped onto $\underline{N}(\underline{\alpha}_p)$.

M3) The vertices 1,2 of $V(G)$ are mapped to the vertices α_{p+1} and β_{p+1} in $A(R_n)$ and $B(R_n)$. Recall that these vertices of R_n satisfy:

$$\alpha_{p+1}, \beta_{p+1} \notin \underline{\alpha}_p \cup \underline{N}(\underline{\alpha}_p)$$
and
$$(\alpha_{p+1}, \beta_{p+1}) \in E(R_n)$$

M4) A subset $Match_{12}$ of $NULL - Match_Y$ having size $|\underline{N}(\alpha_{p+1}, \beta_{p+1}) - \{\alpha_{p+1}, \beta_{p+1}\}|$ is mapped onto $\underline{N}(\alpha_{p+1}, \beta_{p+1}) - \{\alpha_{p+1}, \beta_{p+1}\}$.

M5) The remaining vertices of $G(X_n)$ are associated with the unmapped vertices of $V(R_n)$. These form the V_{rem} vertices of Definition(2).

The lower bound on the size of *NULL* and the definition of "Neighbourhood restricted" establish that this mapping can be constructed. Observe that at most $|V_{rem}|-c_1$ edges can be forbidden in the subgraph consisting of the V_{rem} vertices (i.e by setting the edges in Y to 0). For if *Cut* denotes the set of edges in $\{V_{rem} \times V_{rem}\} \cap Y$ then $|Cut| \leq |V_{rem}| - c_1$, since:

$$|V_{rem}| \geq n - 2|\Gamma_1 \cup \Gamma_2| - c_0 - 2k$$
$$|Cut| \leq n/2 - k - c - |\Gamma_1| - |\Gamma_2|$$

Thus:

$$|V_{rem}| - |Cut| \geq n/2 - |\Gamma_1| - |\Gamma_2| - k$$

But: $|\Gamma_1| + |\Gamma_2| \leq$

So the following

E1) All edges between α_{p+1} and $\underline{N}($

E2) All edges between $\{\alpha_{p+1}, \beta_{p+1}\}$

E3) All edges correspond to edges

E4) A *minimal* subgraph of R_n. c.f

E5) All edges co in V_{rem}, which corr between $\underline{N}(\alpha_{p+1}, \beta_{p+1})$

All remaining e Fig(2). Note that e construction.

If the edge (1,2 the edge (1,2) is mis contrary were true, for some $v \in \Gamma_1 \cup \Gamma_2$ ar added by (E1) must would have degree k most k-1. This contr

But: $|\Gamma_1| + |\Gamma_2| \leq n/2 - k - \mathbf{c}$. Hence:

$$|V_{rem}| - |Cut| \geq \mathbf{c} \geq \mathbf{c}_1$$

So the following edges can now be set to 1.

E1) All edges between $\Gamma_1 \cup \Gamma_2$ and $Match_Y$ which correspond to the edges between α_{p+1} and $\underline{N}(\underline{\alpha}_p)$.

E2) All edges between $\{1,2\}$ and $Match_{12}$ which correspond to the edges between $\{\alpha_{p+1}, \beta_{p+1}\}$ and $\underline{N}(\alpha_{p+1}, \beta_{p+1})$ *except for the edge* (1,2).

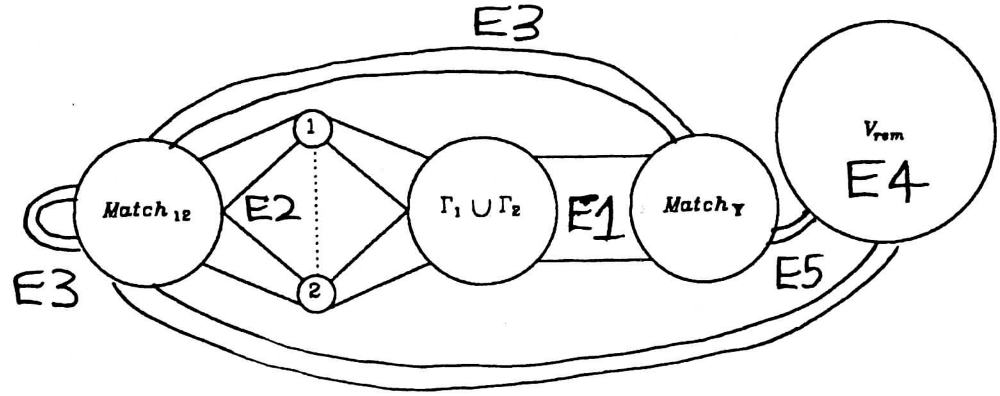
E3) All edges connecting the vertices in $Match_Y$ and $Match_{12}$ which correspond to edges connecting vertices in $\underline{N}(\underline{\alpha}_p)$ and $\underline{N}(\alpha_{p+1}, \beta_{p+1})$.

E4) A *minimal* set of edges in $\{V_{rem} \times V_{rem}\} - Y$ to correspond to the V_{rem} subgraph of R_n. c.f The size of *Cut* and the "extremal number" property of R_n.

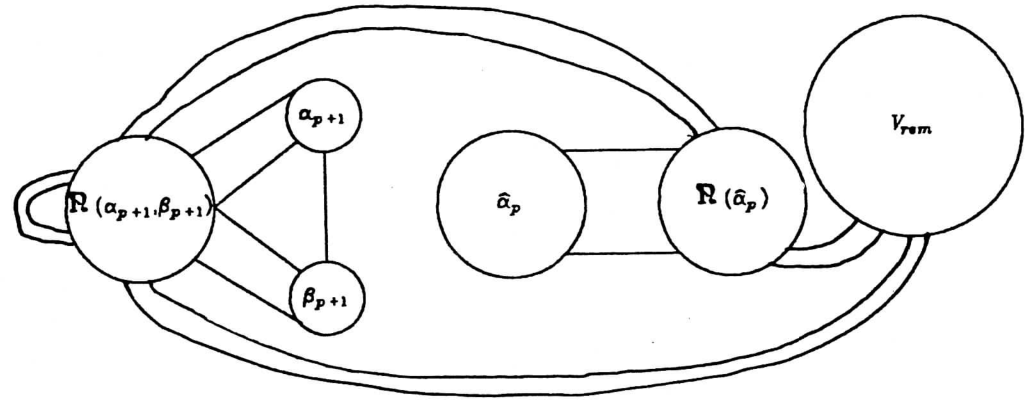
E5) All edges connecting vertices in $Match_D$, where D is Y or 12, to vertices in V_{rem}, which correspond to edges between $\underline{N}(\underline{\alpha}_p)$ and V_{rem} in R_n and edges between $\underline{N}(\alpha_{p+1}, \beta_{p+1})$ and V_{rem} in R_n

All remaining edges in $X_n - Y - (1,2)$ are set to 0. This yields the graph of Fig(2). Note that each of the vertices in *NULL* has degree *exactly* k in this construction.

If the edge (1,2) *is* present in G, then clearly G is an $\underline{R}$-graph. However if the edge (1,2) is missing from G then it cannot be an $\underline{R}$-graph. For suppose the contrary were true, and that F is an R_n-subgraph of G. Since R_n is k-regular, for some $v \in \Gamma_1 \cup \Gamma_2$ an edge $(1,v)$ must be in **E(F)**, and so one of the edges (v,w) added by (E1) must be absent from **E(F)**, where $w \in Match_Y$, for otherwise v would have degree $k+1$ in F. But now the vertex w can now have degree at most $k-1$. This contradiction proves the theorem. □



(a) $G(X_n)$



(b) R_n

Figure 2

1. A.Borodin, D.Do
Proceedings of 15t

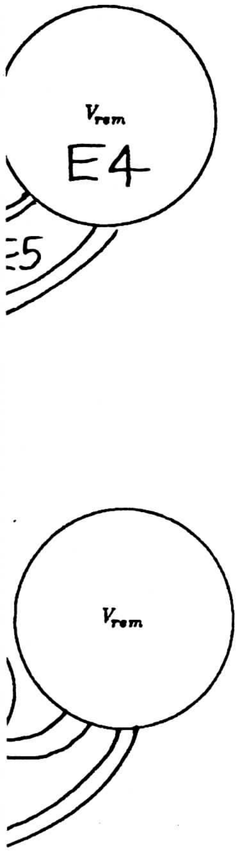
2. A.Cobham. The
on SWAT, 1966, pp.

3. W.Masek. A Fas
Complexity. Maste

4. P.Pudlak. A Lo
1983.

5. I.Wegener. Opti
Boolean Functions.

6. I.Wegener. On t
Functions. Univ. O



REFERENCES

1. A.Borodin, D.Dolev, F.E.Fich, W.Paul. Bounds For Width-2 Branching Programs. Proceedings of 15th STOC, 1983, pp. 87-93.

2. A.Cobham. The Recognition Problem for the Set of Perfect Squares. 7th Symposium on SWAT, 1966, pp. 78-87.

3. W.Masek. A Fast Algorithm for the String Editing Problem and Decision Graph Complexity. Master Th., M.I.T,1976.

4. P.Pudlak. A Lower Bound on the Complexity Of Branching Programs. Univ. Of Prague, 1983.

5. I.Wegener. Optimal Decision Trees and 1-Time Only Branching Programs for Symmetric Boolean Functions. Proc. 9th CAAP, 1984.

6. I.Wegener. On the Complexity of Branching Programs and Decision Trees for Clique Functions. Univ. Of Frankfurt, May, 1984.