Properties of Random VAFs and Implications for Efficient Algorithms

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Abstract. By gaining insight into the structure and behaviours of objects drawn at random from a general class, it is often possible to develop algorithms and techniques which ameliorate the computational difficulty of decision questions arising in the general case. In this paper we present a number of approaches for the random generation of *value-based argumentation frameworks* (VAFs) built on n arguments and using k values. Via an empirical study we consider the behaviour of the associated random VAFs with respect to the issue of how many arguments within them have the property of being "*objectively accepted*". Our studies indicate that the property of having no objectively accepted argument exhibits a so-called "phase-transition effect", similar in nature to those observed in many other well-established AI studies.

Keywords. Computational properties of argumentation; Value-based Argumentation Frameworks; randomized algorithms

Introduction

An understanding of the characteristics underlying typical instances of a computational problem may provide useful insight into the development of feasible algorithmic methods. As a result, even notionally intractable problems may be found to have efficient *average-case* solutions, provided that the most demanding instances for a given algorithm are comparatively rare. A well-known example of this phenomenon is deciding if a propositional formula, presented in CNF, is satisfiable. Despite its status as a canonical NP–complete problem, as shown by Wu & Tang [1], this has an average-case polynomial time algorithm (under one of the standard models for random instances of CNF formulae involving n propositional variables, i.e. that of Goldberg [2]).

There has been considerable recent activity dedicated to aspects of algorithmic methods applied to the abstract argumentation framework (AF) formalism proposed by Dung [3], almost without exception this has engaged with identifying so-called "tractable fragments": that is, special cases which admit efficient worst-case algorithms for some decision problems. Thus, Dung [3] already identifies *acyclic* AFs as such a fragment, Dunne [4] (bipartite AFs) and Coste-Marquis *et al.* [5] (symmetric AFs) extend the range of classes. Recently more general mechanisms exploiting "tree-width" and "clique-width" parameters have been shown to offer promising characteristics in work of Ordnyiak & Szeider [6], and Dvořák *et al.* [7,8]. In contrast, however, the potential use of average-case solution techniques has been largely underdeveloped.

There are, of course, a number of reasons one could advance to explain this apparent oversight. Not least among these is the often formidable challenge posed in *formally* demonstrating that a proposed approach is *indeed* effective: average-case analyses tending to be significantly more demanding that their worst-case counterparts. In order to address such issues one can consider providing supporting evidence via *empirical* approaches. Recent work of Atkinson *et al.* [9] describes a suite of methods through which *random instances* of divers forms of AFs may be generated. Among these, in addition to standard AFs are value-based frameworks (VAF) as described in Bench-Capon [10], and *extended* AFs as proposed in Modgil [11]. Such approaches offer a basis upon which empirical studies can be constructed and the behaviour of "typical" frameworks assessed.

Our aim in this paper is to consider a number of properties of random VAFs – both under uniform allocation of values to arguments and within two "biased" methods. Our results offer strong empirical support that a number of known intractable decision problems regarding VAFs in fact may be solved by average-case polynomial time algorithms. In the remainder of this paper we review background definitions in Sect. 1 and describe a number of models of random VAFs in Sect. 2. In Sect. 3 we present empirical evidence regarding the characteristics of random VAFs with respect to a number of specific properties. Conclusions and directions for further work are offered in Sect. 5.

Prior to presenting our results, however, we briefly motivate the choice of Bench-Capon's VAF abstraction[1] as of particular importance from the viewpoint of developing efficient average-case solutions. The VAF model offers two significant features of interest. Firstly, from the perspective of "random generation methods", as discussed in detail in Atkinson *et al.* [9] and summarised in Sect. 2, unlike standard AFs which may for such purposes be considered as directed graph structures, a number of non-trivial issues arise in regard to both *enumeration* and random generation matters. Secondly, again in contrast to classical AFs, until quite recently no useful tractable fragments of VAFs had been identified.[2] Thus, as shown by Dunne [4], graph structures as basic as binary trees (even under the condition that no value is common to three or more arguments) fail to reduce the computational complexity of standard VAF decision questions from the NP, coNP–complete status of the unrestricted case. Given this situation, developing efficient average-case methods would be particularly beneficial.

1. Background.

The following concepts were introduced in Dung [3].

Definition 1 *An* argumentation framework *(*AF*) is a pair $\mathcal{H} = \langle \mathcal{X}, \mathcal{A} \rangle$, in which $\mathcal{X}$ is a finite set of* arguments *and $\mathcal{A} \subseteq \mathcal{X} \times \mathcal{X}$ is the* attack relationship *for $\mathcal{H}$. A pair $\langle x, y \rangle \in \mathcal{A}$ is referred to as 'y* is attacked by *x' or 'x* attacks *y'. $x \in \mathcal{X}$ is* acceptable with respect to *S if for every $y \in \mathcal{X}$ that attacks x there is some $z \in S$ that attacks y; S is* conflict-free *if no argument in S is attacked by any other argument in S. A conflict-free set S is* admissible *if every $y \in S$ is acceptable w.r.t S; S is a* preferred extension *if it is a*

[1] It has been claimed that the semantics operating in VAFs may lead to inconsistent conclusions. Although, to the authors' knowledge this is an open question, we note that the alleged demonstration of "inconsistency" from [12] is comprehensively refuted in [13].

[2] The first such fragments are presented in work of Dunne [14] and Kim *et al.* [15].

maximal (with respect to $\subseteq$) admissible set; An argument x is credulously accepted *if there is* some *preferred extension containing it; x is* sceptically accepted *if it is a member of* every *preferred extension.*

Bench-Capon [10] develops the concept of "attack" from Dung's model to take account of values and thereby distinguish notions of attack from *successful* attack.

Definition 2 *A* value-based argumentation framework *(*VAF*), is defined by a triple $\mathcal{H}^{(\mathcal{V})} = \langle \mathcal{H}(\mathcal{X}, \mathcal{A}), \mathcal{V}, \eta \rangle$, where $\mathcal{H}(\mathcal{X}, \mathcal{A})$ is an* AF*, $\mathcal{V} = \{v_1, v_2, \ldots, v_k\}$ a set of k* values*, and $\eta : \mathcal{X} \to \mathcal{V}$ a mapping that associates a value $\eta(x) \in \mathcal{V}$ with each argument $x \in \mathcal{X}$. An* audience *for a* VAF *$\langle \mathcal{X}, \mathcal{A}, \mathcal{V}, \eta \rangle$, is a binary relation $\mathcal{R} \subset \mathcal{V} \times \mathcal{V}$ whose (irreflexive) transitive closure, $\mathcal{R}^*$, is asymmetric, i.e. at most one of $\langle v, v' \rangle$, $\langle v', v \rangle$ are members of $\mathcal{R}^*$ for any distinct v, $v' \in \mathcal{V}$. We say that v_i* is preferred to *v_j* in the audience *$\mathcal{R}$, denoted $v_i \succ_{\mathcal{R}} v_j$, if $\langle v_i, v_j \rangle \in \mathcal{R}^*$. We say that α is a* specific *audience if α yields a* total *ordering of $\mathcal{V}$. The notation $\mathcal{U}$ is used for the set of all specific audiences over $\mathcal{V}$*

A standard assumption from [10] which we retain in our subsequent development is the following:
Multivalued Cycles Assumption (MCA)
For any *simple cycle* of arguments in a VAF, $\langle \mathcal{X}, \mathcal{A}, \mathcal{V}, \eta \rangle$, – i.e. a finite sequence of arguments $y_1 y_2 \ldots y_i y_{i+1} \ldots y_r$ with $y_1 = y_r$, $|\{y_1, \ldots, y_{r-1}\}| = r - 1$, and $\langle y_j, y_{j+1} \rangle \in \mathcal{A}$ for each $1 \leq j < r$ – there are arguments y_i and y_j for which $\eta(y_i) \neq \eta(y_j)$.

In less formal terms, this assumption states every simple cycle in $\mathcal{H}^{(\mathcal{V})}$ uses at least two distinct values.

Using VAFs, ideas analogous to those introduced in Defn. 1 are given by relativising the concept of "attack" using that of *successful* attack with respect to an audience. Thus,

Definition 3 *Let $\langle \mathcal{X}, \mathcal{A}, \mathcal{V}, \eta \rangle$ be a* VAF *and $\mathcal{R}$ an audience. For arguments x, y in $\mathcal{X}$, x is a* successful attack *on y (or x* defeats *y)* with respect to the audience *$\mathcal{R}$ if: $\langle x, y \rangle \in \mathcal{A}$* and *it is* not *the case that $\eta(y) \succ_{\mathcal{R}} \eta(x)$.*

Replacing "attack" by "successful attack w.r.t. the audience $\mathcal{R}$", in Defn. 1 yields definitions of "conflict-free", "admissible set" etc. relating to value-based systems, e.g. S is conflict–free w.r.t. to the audience $\mathcal{R}$ if for each x, y in S it is not the case that x successfully attacks y w.r.t. $\mathcal{R}$. It may be noted that a conflict-free set in this sense is not necessarily a conflict-free set in the sense of Defn. 1: for x and y in S we may have $\langle x, y \rangle \in \mathcal{A}$, provided that $\eta(y) \succ_{\mathcal{R}} \eta(x)$, i.e. the value promoted by y is preferred to that promoted by x for the audience $\mathcal{R}$.

Bench-Capon [10] proves that every specific audience, α, induces a unique preferred extension within its underlying VAF: for a given VAF, $\mathcal{H}^{(\mathcal{V})}$, we use $P(\mathcal{H}^{(\mathcal{V})}, \alpha)$ to denote this extension. Analogous to the concepts of credulous and sceptical acceptance, in VAFs the ideas of *subjective* and *objective* acceptance arise.
Subjective Acceptance (SBA)
Instance: $\mathcal{H}(\mathcal{X}, \mathcal{A}, \mathcal{V}, \eta)$ and $x \in \mathcal{X}$.
Question: Is there a specific audience, α, for which $x \in P(\langle \mathcal{X}, \mathcal{A}, \mathcal{V}, \eta \rangle, \alpha)$?
Objective Acceptance (OBA)
Instance: $\mathcal{H}(\mathcal{X}, \mathcal{A}, \mathcal{V}, \eta)$ and $x \in \mathcal{X}$.
Question: Is $x \in P(\langle \mathcal{X}, \mathcal{A}, \mathcal{V}, \eta \rangle, \alpha)$ for *every* specific audience α?

As we have noted earlier, the complexity of SBA (NP–complete) and OBA (coNP–complete) is known to be unchanged under quite extreme restrictions on the form of instances as shown in Dunne [4].

2. Models of Random VAFs.

In Atkinson *et al.* [9] several distinctive features concerning "uniform generation methods" (that is, those for which, given n and k, each n argument VAF using exactly k values is equally likely to be reported) are identified and discussed. The basic algorithm presented is reproduced as Algorithm 1.

Algorithm 1 Random Generation of VAFs (from [9])

1: **Input:** $\langle n, k, p\rangle$ ($p \in [0, 1]$, $n \geq 0$, $1 \leq k \leq n$)
2: $\mathcal{X} := \{x_1, \ldots, x_n\}$;
3: $\mathcal{X}_i := \emptyset$ **for** $1 \leq i \leq k$;
4: $\mathcal{A} := \emptyset$;
5: $n_i := 0$ **for** $1 \leq i \leq k$;
6: $\mathcal{V} := \{v_1, v_2, \ldots, v_k\}$;
7: $\eta :=$ Random mapping of $\mathcal{X}$ to $\mathcal{V}$; //* Implementation discussed in sequel. *//
8: **for** $i = 1$; $i \leq k$; $i++$ **do**
9:     $\mathcal{X}_i := \{x \in \mathcal{X} : \eta(x) = v_i\}$;
10:     $\langle \mathcal{X}_i, \mathcal{A}_i\rangle :=$ Random *acyclic* AF;
11:     $\mathcal{A} := \mathcal{A} \cup \mathcal{A}_i$;
12: **end for**
13: **for** $i = 1$; $i \leq k$; $i++$ **do**
14:     **for** $j = i + 1$; $j \leq k$; $j++$ **do**
15:         **for** each $s \in \mathcal{X}_i$ **do**
16:             **for** each $t \in \mathcal{X}_j$ **do**
17:                 $b :=$ Uniformly randomly chosen real value in the interval $[0, 1)$;
18:                 **if** $b < p$ **then**
19:                     $\mathcal{A} := \mathcal{A} \cup \{\langle s, t\rangle\}$;
20:                 **end if**
21:                 $b :=$ Uniformly randomly chosen real value in the interval $[0, 1)$;
22:                 **if** $b < p$ **then**
23:                     $\mathcal{A} := \mathcal{A} \cup \{\langle t, s\rangle\}$;
24:                 **end if**
25:             **end for**
26:         **end for**
27:     **end for**
28: **end for**
29: **return** $\langle\langle \mathcal{X}, \mathcal{A}\rangle, \mathcal{V}, \eta\rangle$;

A significant factor in the nature of the output reported by this algorithm concerns the implementation of (l. 7). For our experimental studies we consider *two* distinct approaches.

VM1. The mapping $\eta : \mathcal{X} \rightarrow \mathcal{V}$ is formed by

a. Uniformly at random choosing a *partition*, $\alpha = \langle a_1, a_2, \ldots, a_k\rangle \in \mathbb{N}^k$, of n into k non-increasing and non-zero parts. That is, for which $a_i \geq a_{i+1} \geq 1$ (for each $1 \leq i < k$) and $\sum_{i=1}^{k} a_i = n$.
b. Uniformly at random choosing a_i arguments from $\mathcal{X}$ whose value will be fixed to v_i.

VM2. For each $x \in \mathcal{X}$ in turn set $\eta(x) = v$ in such a way that $P[\eta(x) = v] = 1/|\mathcal{V}|$.

We note that the rationale supporting the first of these methods (VM1) is that (when p in Algorithm 1 is fixed at 0.5) this approach is *uniform* whereas VM2 fails to be so. While the reasons are discussed more fully in [9], this arises from the fact that for the purposes of uniform generation the significant factor is the relative *numbers* of arguments with a given value and not *which specific* arguments these are. We note that while the expected number of arguments with any given value using VM2 is $\approx |\mathcal{X}|/|\mathcal{V}| = n/k$, this is *not* the case w.r.t. (VM1). The main problem with (VM1) is that its implementation, even for moderate values of n and k, can be prohibitively slow. As we shall outline in the following section, however, in terms of VAF characteristics it turns out that (VM1) and (VM2) have similar behaviour.

We shall subsequently use $\mathcal{H}_{n,k,p}^{\mathrm{VM}}$ (where VM is one of (VM1)–(VM2) above) to denote the random variable defined by the output of Alg. 1 given input $\langle n, k, p\rangle$ and constructing the value mapping, η via the method specified in VM.

3. Properties of "almost all" VAFs.

Using Alg. 1 configured using each of the methods (VM1)–(VM2) to determine the value mapping η, we examine average-case behaviours in relation to the function, $p_{\mathrm{OBA}}^{\mathrm{VM}} : \mathbb{N} \times \mathbb{N} \times [0,1] \to [0,1]$ given by

$$p_{\mathrm{OBA}}^{\mathrm{VM}}(n,k,p) \quad = \quad \mathit{Prob}[\mathcal{H}_{n,k,p}^{\mathrm{VM}} \text{ has at least one \emph{objectively accepted} argument.}]$$

We observe that, in principle, one is dealing with *four* quantities: the triple $\langle n, k, p\rangle$ and the outcome being analysed. In order to present outcomes in an accessible *graphical* form, however, it is useful to note that by considering the single value $n^2 + k$ we are able to combine n and k without loss of information. Recalling that $1 \leq k \leq n$, the value $n^2 + k$ uniquely determines $\langle n, k\rangle$: given $m = n^2 + k$, n is recovered by $\lfloor\sqrt{m}\rfloor$, and (consequently) k via $m - (\lfloor\sqrt{m}\rfloor)^2$.

The following experiment was carried out:

Experiment: This considered each $n \in \{100, 150, 200, 250, 300, 350, 400\}$ and $6 \leq k \leq 20$. For the value partition method VM2 all $105 = 7 \times 15$ pairs $\langle n, k\rangle$ were examined, while with VM1 only the 18 cases from $n \leq 350$ and $6 \leq k \leq 8$.[3] In each case, 100 VAFs were sampled from $\mathcal{H}_{n,k,p}^{\mathrm{VM}}$ for each p, the range of p being defined in terms of multiples of $\log(n+k)/(200(n+k)$ (where $\log$ is the natural logarithm). These multiples (m) label the x-axis of the various output plots given below.

[3] The value mapping mechanism required for VM1, involves a random partition generation algorithm whose run-time is from $O(n^{k+1})$. Background to this is discussed in Atkinson *et al.*[9].

In each of the 100 trials with a fixed $\langle n, k, p \rangle$ the number of occasions in which at least one objectively accepted argument was "discovered" was recorded. Given that it is infeasible exhaustively to enumerate all $k!$ specific audiences for larger values of k, a randomly chosen sample of k^2 ordering were chosen so that an argument was reported as "objectively accepted" if every such audience led to its acceptance.[4] The y-axis describes the proportion $S/100$ where S is the number of cases in 100 instances which report at least one OBA argument. For reasons of space we concentrate on results for VM2 noting that VM1 has similar (though not identical) characteristics.

In Figure 1 the effect of varying n with $k = 20$ is shown, while Figure 2 illustrates the behaviour resulting for $n = 400$ and k varying.

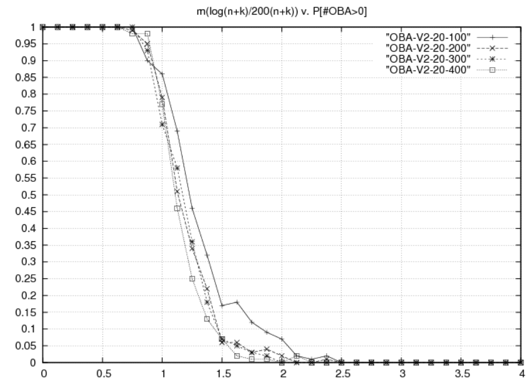

Figure 1. Objective Acceptance – VM2 behaviour with $k = 20$ and $n \in \{100, 200, 300, 400\}$

As is seen in Figure 1, the transition from "almost every" to "almost none" becomes more pronounced as n increases for a fixed k, with a similar effect as k increases being noticeable in Figure 2

The behaviour of VM1 and VM2 lends support to the following,

Conjecture 1 *For* $X \in \{VM1, VM2\}$*, for all* $k \in \mathbb{N}$*, there is a* positive constant, $\theta_k^X \in \Re^+$ *for which*

$$\lim_{n \to \infty} p_{\mathrm{OBA}}^X \left(n, k, \frac{\alpha \log n}{n} \right) = 0 \ \textit{if}\ \alpha > \theta_k^X$$

$$\lim_{n \to \infty} p_{\mathrm{OBA}}^X \left(n, k, \frac{\alpha \log n}{n} \right) = 1 \ \textit{if}\ \alpha < \theta_k^X$$

[4]Note that whereas there is a small chance of arguments being incorrectly reported as objectively accepted, whenever an argument is reported as *not* having this property, such reports are guaranteed to be accurate.

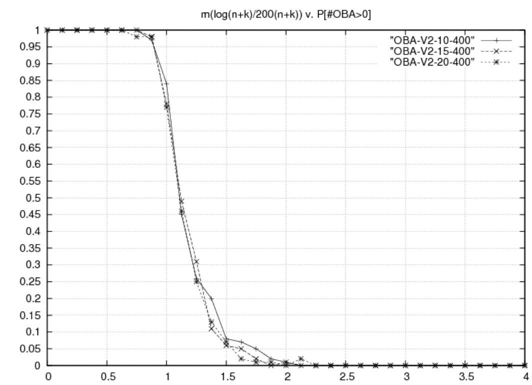

Figure 2. Objective Acceptance – VM2 behaviour with $n = 400$ and $k \in \{10, 15, 20\}$

Notice that the outcomes from the experiments suggest a region for θ_k^X of

$$0.00375 < \theta_{VM1} < 0.02875 \ ; \ 0.00375 < \theta_{VM2} < 0.02$$

(Recall that the labelling of the x-axis in these figures defines *multiples* of $1/200$.)

4. Towards an Analytic Verification of Empirical Behaviour.

Consider the value partition method defined through VM2: that is, in which each argument is allocated a specific value, v_i, with probability $1/|\mathcal{V}|$ independently of other choices made. We state the following properties without proof (for space reasons).

Lemma 1 *Let $D_{n,p}(\mathcal{X}, \mathcal{A})$ be the random variable corresponding to an* acyclic AF *with n arguments $\mathcal{X}$ and attacks included independently with probability p. Let $g(n,p)$ denote the* expected size *of the grounded extension of $D_{n,p}$. Then $g(n,p)$ satisfies*

$$g(1,p) \ = \ 1 \ ; \ g(n,0) \ = \ n \ ; \ g(n,1) \ = \ 1$$

$$g(n,p) \ = \ \frac{1-(1-p)^n}{p} \ + \ g\left(\left(n - \frac{1-(1-p)^n}{p}\right)\left((1-p)^{\frac{1-(1-p)^n}{p}}\right), \ p\right)$$

Lemma 2 $$g\left(n, \frac{\alpha \log n}{n}\right) \ \approx \ \frac{n}{\alpha \log n}\left(1 \ - \ \frac{1}{n^\alpha}\right)$$
and, for fixed p, $g(n,p) \ \approx \ C_p \log n$, with C_p dependent on p.

Coupling Lemma 1 with the following property of objectively accepted arguments suggests not only an approach to analytically confirming existence issues for these within VAFs but also hints at a potentially fast *average-case* algorithm for deciding whether an argument is *not* so accepted.

Lemma 3 *Let $\langle \mathcal{X}, \mathcal{A}, \mathcal{V}, \eta \rangle$ be any* VAF *and x an argument in $\mathcal{X}$. If x is objectively acceptable then no attacker y of x belongs to $G_{\eta(y)}$.*

Hence, via Lemma 3, a basic algorithm for deciding $\neg$OBA, is: test if $y \in G_{\eta(y)}$ for each $y \in \{x\}^-$, reporting $\neg\text{OBA}(\mathcal{H}, x)$ if any such y is found.

5. Conclusions and development.

The focus of this paper has reviewed random bases for constructing VAFs. The long term aim of this approach is to exploit the behaviour observed in guiding the design of efficient *average-case* algorithms. To illustrate its viability we have presented a preliminary experimental study of one property: the likelihood of an objectively accepted argument being present. The outcomes offer strong indications that so-called "phase-transition behaviours" arise, irrespective of how value mappings are constructed. Overall our interest is in identifying characteristics by which a wide range of VAF problems may be considered, e.g. SBA, counting questions, etc.

There are, of course, a wide variety of proposals for developments of Dung's AF abstraction. The ideas promoted in the current article may well prove fruitful within this more general context and, in principle, add a range of important algorithm design techniques whose applicabilty to the abstract AF sphere has, to date, been barely addressed.

References

[1] L.C. Wu and C.Y. Tang. Solving the satisfiability problem by using randomized approach. *Inf. Proc. Letters*, 41:187–190, 1992.

[2] A. Goldberg. Average case complexity of the satisfiability problem. In *Proc. 4th Workshop on Automated Deduction*, pages 1–6, 1970.

[3] P. M. Dung. On the acceptability of arguments and its fundamental role in nonmonotonic reasoning, logic programming, and N-person games. *Artificial Intelligence*, 77:321–357, 1995.

[4] P. E. Dunne. Computational properties of argument systems satisfying graph-theoretic constraints. *Artificial Intelligence*, 171:701–729, 2007.

[5] S. Coste-Marquis, C. Devred, and P. Marquis. Symmetric argumentation frameworks. In *Proc. 8th ECSQARU*, volume 3571 of *LNAI*, pages 317–328. Springer-Verlag, 2005.

[6] S. Ordyniak and S. Szeider. Augmenting tractable fragments of abstract argumentation. In *Proc. 22nd IJCAI*, pages 1033–1038. AAAI Press, 2011.

[7] W. Dvořák, S. Szeider, and S. Woltran. Reasoning in argumentation frameworks of bounded clique-width. In *Proc. 3rd COMMA*, volume 216 of *FAIA*, pages 219–230. IOS Press, 2010.

[8] W. Dvořák, R. Pichler, and S. Woltran. Towards fixed-parameter tractable algorithms for abstract argumentation. *Artificial Intelligence*, 186:1–37, 2012.

[9] K. Atkinson, P. E. Dunne, and S. Nofal. Random generation methods for classes of argumentation framework. *(in submission)*, 2013.

[10] T. J. M. Bench-Capon. Persuasion in Practical Argument Using Value-based Argumentation Frameworks. *Journal of Logic and Computation*, 13(3):429–448, 2003.

[11] S. Modgil. Reasoning about preferences in argumentation frameworks. *Artificial Intelligence*, 173(9–10):901–934, 2009.

[12] M. Caminada and Y. Wu. On the limitations of abstract argumentation. In *Proc. 23rd BNAIC*, pages 59–66, 2011.

[13] S. Nofal, K. Atkinson, and P. E. Dunne. Algorithms for decision problems in argument systems under preferred semantics. *Artificial Intelligence*, 207:23–51, 2014.

[14] P. E. Dunne. Tractability in value-based argumentation. In *Proc. 3rd COMMA*, volume 216 of *FAIA*, pages 195–206. IOS Press, 2010.

[15] E. J. Kim, S. Ordyniak, and S. Szeider. Algorithms and complexity results for persuasive argumentation. *Artificial Intelligence*, 175:1722 – 1736, 2011.