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MODAL DECISION PROBLEMS

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1 MODAL LOGIC AS 'DIE KLASSENTHEORIE'

There are different views on the subject of Modal Logic. For the purpose of this chapter it is important to distinguish between two of them.

According to the *local* view, Modal Logic deals with a number of *concrete modal logics*. Since the beginning of the 20th century developers and users of Modal Logic from philosophy, mathematics, computer science, artificial intelligence, linguistics and other fields have introduced and investigated dozens of particular modal logics suitable for their

needs: *epistemic*, *provability*, *temporal*, *dynamic*, *description*, *spatial*, to mention just a few.

With the number of concrete modal logics introduced in the literature growing, there came an understanding that it may be interesting and important to formulate general abstract notions of modal logics and to investigate the landscape of the resulting classes of logics and their properties. The pioneers of this *global* approach were Scroggs [127] who considered all extensions of **S5**, Dummett and Lemmon [33] who studied all logics between **S4** and **S5**, Bull [14] and Fine [40] who investigated the logics containing **S4.3**, and Lemmon [86, 87, 88] and Segerberg [129] who launched a systematical investigation of various classes of modal logics. Two other influential figures that should also be mentioned here are Kuznetsov [81, 84, 85] and Jankov [67, 66, 68, 69] who investigated the class of all extensions of intuitionistic propositional logic which is closely related to the class of modal logics containing **S4**; see Section 9.

Although not formulated explicitly, the 'globalist's' dream research programme was to develop a mathematical machinery that could provide general solutions to the following major problems:[1]

1. given a class of models/structures, axiomatise the modal logic it determines, decide in an effective way whether it has certain important properties, say, decidability, compactness, interpolation, etc., and determine its computational complexity.

2. given a modal logic in the form of a finite set of axioms and inference rules, characterise the (simplest, smallest, largest, etc.) class of models/structures with respect to which this logic is sound and complete, decide in an effective way whether it has important properties as above, and determine its computational complexity.

This research programme is formulated in quite general terms and therefore can be interpreted in various ways. For example, it is not specified what kind of classes of frames/models we consider and what kind of axiomatic systems we take into account. Of course, different interpretations may lead to different solutions, but anyway first results within this ambitious programme looked very promising indeed! For example, Bull [14] proved that all extensions of **S4.3** have the finite model property and Fine [40] showed that all of them are finitely axiomatisable, and so decidable. (Actually, Dummett and Lemmon [33] claimed that all logics between **S4** and **S5** have the finite model property, but their proof was wrong: ten years later Jankov [68] constructed a counterexample.) In view of Makinson's theorem [94], one can effectively decide whether a given logic above **K** is consistent. Maksimova [95, 97] proved that two properties of logics containing **S4**—tabularity and interpolation—are decidable as well. It seems that many modal logicians did believe in an eventual success of this Big Programme.

In this chapter we analyse the development of Modal Logic within the research framework formulated above, starting from the beginning of the 1970s, although not necessarily in chronological order; for a historical analysis of mathematical modal logic the reader is referred to the recent paper of Goldblatt [57] and notes in [24]. Because of space limitations, we mainly concentrate on normal (multi-) modal logics and their decidability and completeness (in particular, with respect to Kripke or finite frames).

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[1]Kuznetsov did formulate such problems *explicitly* in the context of superintuitionistic logics; e.g., given an axiomatisation of a superintuitionistic logic, can we decide in an effective way whether the logic is characterised by a finite algebra?

Roughly, our plan is as follows. We start in Section 2 with Thomason's explication (i′) of the *semantical* part (i) of the research programme above. Then, in Section 3, we lay the foundation for the most important *syntactical* notion of Modal Logic, namely, that of a *normal modal logic*. Having introduced an adequate semantics for normal modal logics in terms of *general frames*, we discuss in detail Blok's dichotomy in order to clarify the difference between Thomason's semantical definition of modal logics and the syntactically defined normal modal logics. Based on this discussion, we then come to the appropriate refinement (ii′) of the *syntactical* part (ii) of the research programme for normal modal logics and solutions to it given by Chagrov and Thomason.

Although beautiful from a mathematical point of view, the results of Thomason and Chagrov are 'negative' in the sense that almost all general algorithmic problems formulated in the Big Research Programme turn out to be undecidable. In the same way as the negative solution to the *classical decision problem* of Hilbert transformed the original problem into a *classification* problem, the 'negative' solution to the modal decision problems brings us down to a more 'modest' and realistic 'relativisation' of the programme to various syntactically or semantically defined classes of modal logics.

In Section 4, we consider logics axiomatised by formulas satisfying certain syntactical constraints, in particular, Sahlqvist formulas, uniform formulas, modal reduction principles, etc., and see whether such constraints allow us to prove general decidability/completeness results. In Section 5, we survey the literature on general decidability/completeness results for logics with some 'strong' axioms, say, extensions of tabular and pretabular logics, logics of finite depth and width, extensions of **S4.3**, **K5**, etc.

Then, in Section 6, we discuss an attempt to attack the Big Research Programme for normal extensions of **K4** (that is, unimodal logics with transitive general frames) and the tense logic **Lin** (of linear flows of time) by means of finite representations of modally definable classes of frames via frame and subframe formulas of Jankov and Fine [69, 41, 45] and more general 'canonical' formulas of [172, 174, 163]. This technique will be also used to draw and discuss connections between extensions of **S4** and superintuitionistic logics in Section 9.

In Section 7 we provide a 'positive' solution to the Big Research Programme for the class of all tense logics of linear flows of time. In fact, it turns out that for this class of logics all the questions posed in the programme are decidable (sometimes even in nondeterministic polynomial time).

In Section 8, we consider the class of subframe logics—i.e., logics determined by classes of (general) frames closed under the formation of substructures in the standard model-theoretic sense—and explore to what extent the research programme can be realised for this semantically defined class of modal logics.

A number of important open problems are formulated throughout the chapter.

2 THOMASON'S ANALYSIS

As we saw in Chapter 1, the standard propositional modal language with a countably infinite set of propositional variables (say, $p_0, p_1, \dots$), the Boolean connectives $\wedge$, $\neg$ (and their derivatives $\rightarrow$, $\vee$, etc.) and unary modal operators $\Box_1, \dots, \Box_n$ can be regarded as a basic tool for talking about relational structures $\mathfrak{F} = \langle W, R_1, \dots, R_n \rangle$, where the R_i are binary relations on $W \neq \emptyset$. We denote this *n-modal language* by $\mathcal{ML}_n$ and call $\mathfrak{F}$ an *n-frame* or simply a (*Kripke*) *frame*, if n is understood.