Sequent calculi for logics with common knowledge (Extended Abstract)

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1 Introduction

Epistemic logics with common knowledge [3, 4] are important for specifying states of multi-agent systems. Likewise, dynamic epistemic logics [2] are used for specifying state transitions in multi-agent systems. Semantically correct analyses of such systems have been given; proof tools for reasoning about agents' knowledge are urgently required. Although there are Hilbert-style formalisations of such logics [3, 4], and sequent calculi for multi-agent modal logics without common knowledge [1], we have found no analytic proof calculi for logics with common knowledge elsewhere. We expect our work to relate to calculi for PDL, to modal μ-calculus and to logics for knowledge and time.

As a step towards automated reasoning in logics with common knowledge, we consider the problems of development of sequent calculi for n-agent modal logics with common knowledge (for simplicity in the case $n = 2$ and for the modal logic $\mathbf{K}$). Admissibility of cut is not yet proved.

2 The sequent calculus $\mathbf{G.K}_2^{\mathbf{C}}$

We adopt approximately the notations of [5]. From the various calculi for zero-order classical logic we choose that of Kleene, **GKc**, in which the rules of *Weakening* and *Contraction* are absorbed into the logical rules.

For modal logic we adopt the notations of [3]. Names for the modal rules indicate which operators are introduced and whether on the left or on the right of $\Rightarrow$. For C read $C_{\{1,2\}}$. In the rule $\mathsf{L_iC}$ ('Left' 'Common knowledge') one may see the Hilbert-style axiom schema $\vdash C\varphi \supset K_i\varphi$ for agent $i \in \{1, 2\}$. In the rule Ind one may see the induction rule ([3] p. 67) "if $\varphi \supset E(\varphi \wedge \psi)$ then $\varphi \supset C\psi$", where E is the general knowledge operator.

Our modal rules are as follows:

$$\frac{C\Gamma', \Gamma \Rightarrow \varphi}{\Gamma'', C\Gamma', K_i\Gamma \Rightarrow K_i\varphi, \Delta}\ \mathsf{RK_i}$$

$$\frac{\Gamma, K_i\varphi, C\varphi \Rightarrow \Delta}{\Gamma, C\varphi \Rightarrow \Delta}\ \mathsf{L_iC} \qquad \frac{\Gamma \Rightarrow \varphi}{\Gamma', C\Gamma \Rightarrow C\varphi, \Delta}\ \mathsf{RC}$$

$$\frac{\Gamma, \varphi \Rightarrow K_1(\varphi \wedge \psi), C\psi, \Delta \quad \Gamma, \varphi \Rightarrow K_2(\varphi \wedge \psi), C\psi, \Delta}{\Gamma, \varphi \Rightarrow C\psi, \Delta}\ \mathsf{Ind}$$

In each case it should be clear what the *principal formula* occurrence is; we also designate the occurrences of formulae in $C\Gamma'$, $K_i\Gamma$ of $\mathsf{RK_i}$ as *semi-principal*, and similarly for the formulae in $C\Gamma$ of RC.

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An early version of this calculus omitted the $C\Gamma'$ part of the $\mathsf{RK_i}$ rule and included a rule $\mathsf{RK_iC}$ allowing the inference of $K_iC\varphi$ from $C\varphi$; this led to immediate problems with cut-elimination, hence the introduction of the extra part of $\mathsf{RK_i}$, which avoided the problems; but then the rule $\mathsf{RK_iC}$ appears to be dispensable, its uses being replaceable by uses of $\mathsf{RK_i}$.

3 Admissibility of structural rules

It is routine to show that all sequents of the form $\Gamma, \varphi \Rightarrow \varphi, \Delta$ are derivable; likewise, admissibility of Weakening and Contraction are routine.

Conjecture 3.1 *The rule*

$$\frac{\Gamma \Rightarrow \varphi, \Delta \quad \varphi, \Gamma \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'} \; Cut$$

is admissible.

Proof.

We argue as usual by induction on the size of the cut-formula and the sum of the heights of the derivations. We present just the cases where the *cut formula* φ is principal (or semi-principal) in both premisses; the usual reductions (the first of which is quite complex) apply to implication and to absurdity. We consider the cases where φ is of the form $K\psi$ or $C\psi$.

1. Cut formula is $K_i\psi$, left derivation ends in $\mathsf{RK_i}$ with cut formula principal, premiss $\Gamma'' \Rightarrow \psi$ and right derivation ends in $\mathsf{RK_i}$ with cut-formula semi-principal and with premiss $\psi, \Gamma''' \Rightarrow \theta$. Reduce this to a cut on ψ with the two premisses, use $\mathsf{RK_i}$ and weaken as required.

2. Cut formula is $C\psi$, left derivation ends in C with cut formula principal, premiss $\Gamma'' \Rightarrow \psi$ and right derivation ends in C with cut-formula semi-principal and with premiss $\psi, \Gamma''' \Rightarrow \theta$. Reduce this to a cut on ψ with the two premisses, use C and weaken as required.

3. Similarly, all cuts where neither premiss involves the induction axiom are reducible. The problem is with the induction rule Ind .... $\square$

All $\mathbf{K_2^C}$ axioms ([3] p. 67) are derivable; this is routine. Once cut is shown to be admissible, we can also justify the rules such as *modus ponens*, necessitation and the Hilbert-style induction rule.

References

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