FORMULA INSTANTIATION IS DECIDABLE FOR THE MONADIC CLASS

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1. Introduction

Given a first-order formula which expresses existence of an element with particular properties, we may be interested in obtaining this element in an explicit form if this formula is satisfiable. Herbrand's theorem says that for any existential formula $\exists\overline{x}\varphi(\overline{x})$ there exists a finite number of term sequences $\overline{t}_1 \dots \overline{t}_n$, such that for any model at least one of them makes $\varphi(\overline{t})$ true in that model. For some formulas just one term sequence is sufficient.

We consider the following decision problem.

Problem. *Given a formula $\varphi(\overline{x})$ does there exist a sequence of terms $\overline{t}$ in the signature Σ_φ of the formula such that $\varphi(\overline{t})$ is valid.*

For the case of quantifier-free formula $\varphi(\overline{x})$ this problem is known as *Formula Instantiation* problem [3]. It is of Σ_2^p complexity in the case of formulas without equality and is undecidable in the general case of the logic with equality.

We study a generalised version of the problem for classes of formulas with quantifiers. As formula instantiation implies satisfiability, we consider only these classes of prefix vocabulary which are decidable with respect to satisfiability. Here we present the proof of decidability of formula instantiation for the class of monadic formulas, referred to as the Ackermann class, see e.g. [1].

2. Monadic case

Let $\varphi(\overline{x})$ be a formula in prenex normal form with an arbitrary quantifier prefix such that it contains only unary functional and relational symbols; and all the free variables which occur in $\varphi(\overline{x})$ are among those in $\overline{x}$. We consider the formula instantiation problem for the class of such formulas. The main result is the following theorem.

Theorem. *We prove that the formula instantiation problem for the class of monadic formulas is decidable.*

Our approach is based on the reduction to the decision problem in the monadic second-order theory of the infinite binary tree [2].

We prove that the class of monadic formulas $\varphi(x)$ without equality is reducible preserving satisfiability to the class of existential monadic formulas by an extension of the signature with a finite set of constant symobols. We construct a new existential formula $\varphi'(x)$, so that the Herbrand's theorem can be applied to conclude that if the formula is satisfiable then it holds in some Herbrand model. We obtain a positive solution of formula instantiation if and only if there exists a term which makes the formula valid in all Herbrand models.

We restrict the domain of x to the set of terms in the signature Σ_φ of the original formula by adding a second-order constraint $TERM_\varphi(x)$. The constraint expresses that a term t substituted for x should contain only constants and function symbols of the original signature.

Hence the following second-order formula obtained from φ' by substituting set variables Z_i for the corresponding propositional symbols

$$\exists x \forall Z_1 \dots Z_n \exists y_1 \dots y_n \Big(\varphi'(x) \wedge TERM(x) \Big),$$

holds true in the monadic second order theory of infinite binary tree.

Finally we introduce two new function symbols f and g and substitute each appearance of f_i in the formula by fg^i which allows to reduce the Herbrand Universe of the formula under consideration to an infinite binary tree, see [2].

It should be noted that the decision procedure for the general case of the monadic second-order theory of a tree, MS2S is of high complexity. For example, monadic second-order theory of one unary function which is reducible to MS2S has non-elementary complexity [1].

References

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