Verifying orientability of rewrite rules using the Knuth-Bendix order

Konstantin Korovin and Andrei Voronkov

University of Manchester
{korovin,voronkov}@cs.man.ac.uk

We consider two decision problems related to the Knuth-Bendix order (KBO). The first problem is *orientability*: given a system of rewrite rules R, does there exist some KBO which orients every ground instance of every rewrite rule in R. The second problem is whether a given KBO orients a rewrite rule. This problem can also be reformulated as the problem of solving a single ordering constraint for the KBO. We prove that both problems can be solved in polynomial time. The algorithm builds upon an algorithm for solving systems of homogeneous linear inequalities over integers. Also we show that if a system is orientable using a real-valued KBO, then it is also orientable using an integer-valued KBO.

Let us describe our results more formally. Let $\succ$ be any order on ground terms and $l \to r$ be a rewrite rule. We say that $\succ$ *orients* $l \to r$, if for every ground instance $l' \to r'$ of $l \to r$ we have $l' \succ r'$. We write $l \succeq r$ if either $l \succ r$ or $l = r$. There are situations where we want to check if there *exists* a simplification order on ground terms that orients a given system of (possible nonground) rewrite rules. We call this problem *orientability.* Orientability can be useful when a theorem prover is run on a new problem for which no suitable simplification order is known. We consider this problem for the Knuth-Bendix orders [5] on ground terms. We give a polynomial-time algorithm for checking orientability by the KBO.

Theorem 1. *The problem of orientability of rewrite rule systems by the Knuth-Bendix order can be solved in polynomial time.*

A similar problem of orientability by the nonground version of the real-valued KBO was studied in [3] and an algorithm for orientability was given. We prove that any rewrite rule system orientable by a real-valued KBO is also orientable by an integer-valued KBO. This result also holds for the nonground version of the KBO considered in [3]. We also show that some rewrite systems could not be oriented by nonground version of the KBO, but can be oriented by our algorithm. In fact the order based on all ground instances is the greatest simplification order extending the ground KBO to nonground terms. Let us note that if we consider the recursive path orderings (RPO) then orientability problem for systems consisting of a single rewrite rule is NP-hard [8].

The second problem we consider is whether a given instance of the KBO orients a rewrite rule. If $\succ$ is total on ground terms, then the problem of checking if $\succ$ orients $l \to r$ has relation to the problem of solving ordering constraints over $\succ$. Indeed, $\succ$ *does not* orient $l \to r$ if, and only if, there exists a ground instance $l' \to r'$ of $l \to r$ such that $r' \succeq l'$, i.e. if, and only if, the ordering constraint $r \succeq l$ has a solution. This means that any procedure for solving ordering constraints consisting of a single inequality can be used for checking if a given system of rewrite rules is oriented by $\succ$, and vice versa. There are two main classes of term orders used in automated deduction and rewriting: the recursive path orders [4, 2] and the Knuth-Bendix order. For each of the them the problem of solving ordering constraints is NP-complete [10, 9, 6]. However, to check if $\succ$ orients $l \to r$, it is sufficient to check solvability of a *single* ordering constraint $r \succeq l$. This problem is NP-complete for the recursive path orders [1], and therefore the problem of checking if an RPO orients a rewrite rule is coNP-complete.

We show that the problem of solving a single ordering constraint for the KBO can be solved in polynomial time.

Theorem 2. *The problem of solving Knuth-Bendix constraints consisting of a single inequality can be solved in polynomial time.*

Proofs of Theorems 1,2 can be found in [7].

References

1. H. Comon and R. Treinen. Ordering constraints on trees. In S. Tison, editor, *Trees in Algebra and Programming: CAAP'94*, volume 787 of *Lecture Notes in Computer Science*, pages 1–14. Springer Verlag, 1994.
2. N. Dershowitz. Orderings for term rewriting systems. *Theoretical Computer Science*, 17:279–301, 1982.
3. J. Dick, J. Kalmus, and U. Martin. Automating the Knuth-Bendix ordering. *Acta Informatica*, 28(2):95–119, 1990.
4. S. Kamin and J.-J. Levy. Two generalizations of the recursive path ordering. Univ. of Illinois at Urbana-Champaign. Unpublished manuscript, 1980.
5. D. Knuth and P. Bendix. Simple word problems in universal algebras. In J. Leech, editor, *Computational Problems in Abstract Algebra*, pages 263–297. Pergamon Press, Oxford, 1970.
6. K. Korovin and A. Voronkov. Knuth-bendix constraint solving is NP-complete. Preprint CSPP-8, Department of Computer Science, University of Manchester, November 2000.
7. K. Korovin and A. Voronkov. Verifying orientability of rewrite rules using the Knuth-Bendix order. Preprint CSPP-11, Department of Computer Science, University of Manchester, February 2001. to appear in RTA'01.
8. Krishnamoorthy and Narendran. On recursive path ordering. *Theoretical Computer Science*, 40:323–328, 1985.
9. P. Narendran, M. Rusinowitch, and R. Verma. RPO constraint solving is in NP. In G. Gottlob, E. Grandjean, and K. Seyr, editors, *Computer Science Logic, 12th International Workshop, CSL'98*, volume 1584 of *Lecture Notes in Computer Science*, pages 385–398. Springer Verlag, 1999.
10. R. Nieuwenhuis. Simple LPO constraint solving methods. *Information Processing Letters*, 47:65–69, 1993.