Interactive Presentations of Mathematics: A Position Paper

Paul A. Cairns & Jeremy Gow
p.cairns@mdx.ac.uk, j.gow@mdx.ac.uk
Middlesex University
The Burroughs, Hendon
London NW4 4BT

Modern mathematics generally takes on a standard form when presenting proofs. The statements of the proof follow in logical progression from previous statements, theorems or axioms. However, there are well known problems with this format:

1. A lack of context obscures the rationale behind the specific form of definitions and theorems [2]...
2. ...however the context is essentially irrelevant to the logical argument.
3. Too much detail in proof obscures the important insights [6] ...
4. ...but a reader may not be able to fill in details for themselves [4].

In essence, there is a tension between too much or too little detail in a proof and furthermore between too much or too little contextualisation. These tensions arise naturally from the restrictions of fixed, printed text where they are resolved to the best of the author's ability. However, a different level of context and detail may be needed for a particular reader. Additionally, the needs of the reader may change. For instance, a reader may require a detailed proof initially but after digesting the proof would no longer need the details and would look rather for the context and motivations behind the proof.

We hold that hypermedia would be an ideal tool to resolve these tensions so that proofs meet the needs of a wide readership and the dynamic needs of an individual reader. Below, we set forth some initial ideas for presenting individual proofs and suggest areas for further development.

Collapsible Proofs

Lamport proposed a structured style for writing proofs [3]. A proof written in this format recursively consists of a list of key proof steps, each justified by a subproof. The result is a hierarchy of nested lists of proof steps: the higher level steps give the outline of the proof, the lower level steps the details.

The obvious adaptation to hypertext is to hide or reveal the subproofs under the reader's control. This was first done by Grundy [1] for a calculational proof style. However, a major drawback is that key insights may be contained within a hidden subproof — but a reader has no way of knowing this at a glance. We believe that a more sophisticated approach should involve *summarising* the hidden parts. In particular, references to key steps, proof methods or imported results can allow the reader to understand the proof step without seeing the details.

Incorporating Context

Apart from understanding the logical argument of a proof, a reader also needs to understand its significance. This comes from both the rationale for its sufficiency (that is, why a given condition is required) and its relationship to other results and definitions (*viz* the theory to which it belongs). We refer to this is as the context of the proof.

Polya founded a school of problem solving that encourages the use of heuristic thinking based on reasoning about the proof in context [5]. The first step is to thoroughly *understand* a theorem with examples and careful examination of each condition. Next is to *plan* a proof and *execute* it with care. Finally, the solver should *reflect* on how the proof relates to the surrounding theory.

We have used Polya's scheme as a framework for presentational hypertext. Collapsible proofs correspond naturally to the plan and execute steps of the framework. The understand step is used to clarify the statement of a theorem: examples illustrate the theorem but also describe the consequences of dropping conditions from the theorem. The reflect step provides links with other online results and definitions. The advantage of using hypermedia in all of these steps is that the reader can control the which parts of the context or the details that they wish to see.

Future Presentations

Our immediate goal is to combine the dynamic management of proof detail and context in hypertext presentation of a single, substantial body of theory, namely an undergraduate course in topology. The longer term goal is to develop tools that will aid future authors, for example, a markup language for proofs and/or theories. A main feature of such tools must be that they are not a burden to use. Providing a full formal proof, as required by the collapsible proof format, may be too much work for an author. We speculate that a connection between authoring tools and automated proving systems could be a real boon in this area.

Nonetheless, it seems that there is still much that can be done in presenting whole theories. Lakatos issued an imperative to the mathematical community to give the full picture of the creative dialectic between theorems and definitions and the growth of theory through generalisation [2], arguing that this is essential to a deep understanding of a theory. Up until now, his challenge has gone unanswered. Perhaps through designing appropriate hypermedia authoring tools, it may be that we can achieve his high goals.

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References

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